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DEUXIÈME CHAPITRE.

Répartition des pressions après écrasement du charbon.

(Communicated at the meeting of January 27, 1940.)

Annexe 1. Se référant au § 2.

Distributions des pressions dans le massif rocheux, chargeant le charbon jusqu'à la résistance à l'écrasement.

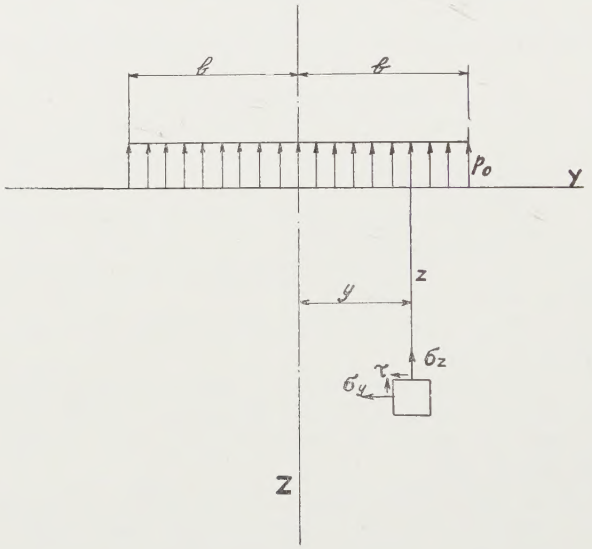

Fig. 18. Hémi-espace sollicité sur une bande de charbon de largeur $2b$ par la tension p_0.

Au lieu d'aborder le problème posé dans la figure 2, nous allons résoudre le problème indiqué en la figure 18 et après nous superposons la pression p_0 règnant dans tous les sens. p_0 est la résistance à l'écrasement du charbon.

En substituant $K = -p_0 dy_0$ dans les formules de MICHELL–FLAMANT de la page 92 du Chapitre I, Proceedings Vol. XLII, No. 2, 1939, on obtient

$$\sigma_z = \frac{2\,p_0\,z}{\pi}\int_{y_0=-b}^{y_0=+b}\frac{dy_0}{\{(y-y_0)^2+z^2\}^2} = \frac{p_0}{\pi}\left\{\frac{(y+b)\,z}{(y+b)^2+z^2} - \right.$$

$$\left. - \frac{(y-b)\,z}{(y-b)^2+z^2} + \operatorname{arc\,tg}\frac{y+b}{z} - \operatorname{arc\,tg}\frac{y-b}{z}\right\}$$

$$\sigma_y = \frac{2\,p_0\,z}{\pi}\int_{y=-b}^{y_0=+b}\frac{(y-y_0)^2\,dy_0}{\{(y-y_0)^2+z^2\}} = \frac{p_0}{\pi}\left\{-\frac{(y+b)\,z}{(y+b)^2+z^2}+\frac{(y-b)\,z}{(y-b)^2+z^2}+\right.$$

$$\left.+\operatorname{arc\,tg}\frac{y+b}{z}-\operatorname{arc\,tg}\frac{y-b}{z}\right\}$$

$$\tau = \frac{2\,p_0\,z^2}{\pi}\int_{y_0=-b}^{y_0=b}\frac{(y-y_0)\,dy_0}{\{(y-y_0)^2+z^2\}^2} = \frac{p_0}{\pi}\left\{\frac{z^2}{(b-y)^2+z^2}-\frac{z^2}{(b+y)^2+z^2}\right\}.$$

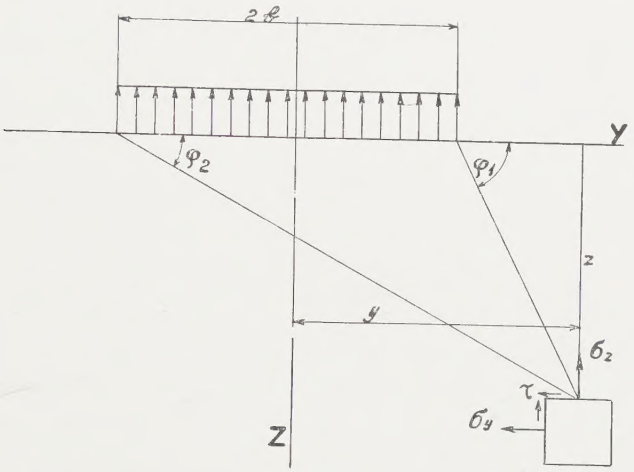

Fig. 19. Même problème que fig. 18. Les points de l'hémi-espace sont fixés par les angles φ_1 et φ_2 au lieu de l'être par les coordonnés y et z.

Quand on se rend compte de la signification des termes, on peut mieux fixer chaque point $y\,z$ par les deux angles φ_1 et φ_2 (Fig. 19) et alors nos formules deviennent comme chez NÁDAI:

$$\sigma_z = \frac{p_0}{2\pi}(2\,\varphi_1 - 2\,\varphi_2 - \sin 2\,\varphi_1 + \sin 2\,\varphi_2) - p_0$$

$$\sigma_y = \frac{p_0}{2\pi}(2\,\varphi_1 - 2\,\varphi_2 + \sin 2\,\varphi_1 - \sin 2\,\varphi_2) - p_0$$

$$\tau = \frac{p_0}{2\pi}(\cos 2\,\varphi_1 - \cos 2\,\varphi_2).$$

Ici nous avons superposé la pression $-p_0$ dans tous les sens pour donner la distribution autour d'une veine de charbon, dont on a exploité une bande (C'est le cas de la figure 2). Nous supposons que la profondeur est telle que des deux côtés le charbon est également écrasé et résiste avec la contrepression p_0.

Annexe 2. Se référant au § 4.

Répartition de la pression sur des piliers de charbon, compte tenu de l'élasticité du charbon et du schiste.

Nous cherchons la répartition de la pression sur un pilier, une bande inexploitée de charbon. (Figure 11). Cette fois pas, comme dans le Chapitre I considérée comme indéformable, mais au contraire élastique et se dilatant transversalement sans résistance, sans frottement, entre toit et mur. Parce que c'est un cas symétrique nous ne considerons que la moitié inférieure. La solution exacte n'est pas connue, mais nous savons que la pression augmente vers les bords et nous posons

$$p = p_0 + k\frac{y^2}{b^2} + l\frac{y^4}{b^4} + m\frac{y^6}{b^6} + \ldots$$

Les coëfficients à déterminer de manière que le dessous du pilier de charbon épouse la surface du mur (le sol).

D'abord nous calculons l'empreinte du mur; pas la profondeur absolue, mais uniquement sa forme. Nous employons une formule donnée dans le Chapitre I, Proceedings Vol. XLII, No. 2, 1939, p. 4.

$$\frac{dw}{dy} = -\frac{2(m^2-1)}{\pi m^2 E}\frac{K}{y}. \quad ^{1})$$

$\frac{dw}{dy}$ est l'inclinaison de la surface due à l'action d'une charge K à une distance y de K. Dans notre cas présent nous remplaçons K par pdy et y par $(y-y_0)$ pour calculer d'abord l'inclinaison de la surface à la distance y du centre. Nous nous contentons de trois termes et trouvons

$$\frac{dw}{dy} = -\frac{2(m^2-1)}{\pi m^2 E}\int_{y_0=-b}^{y_0=+b}\left(p_0 + k\frac{{y_0}^2}{b^2} + l\frac{y_0^4}{b^4}\right)\frac{dy_0}{y_0-y}$$

$$\frac{dw}{dy} = -\frac{2(m^2-1)}{\pi m^2 E}\left[p_0 \ln\frac{b+y}{b-y} + \frac{k}{b^2}\left(y^2 \ln\frac{b+y}{b-y} - 2by\right) + \frac{l}{b^4}\left\{y^4 \ln\frac{b+y}{b-y} - by\left(\tfrac{2}{3}b^2 + 2y^2\right)\right\}\right].$$

Pour $y=0$ $\frac{dw}{dy}=0$.

En intégrant encore une fois on trouve l'empreinte sauf la constante d'intégration (L'index s veut dire que c'est la déformation de la surface du schiste).

$$w_s = \frac{2(m^2-1)}{\pi m^2 E_s}\left\{p_0\left(b \ln\frac{b^2}{b^2-y^2} - y \ln\frac{b+y}{b-y}\right) + \frac{k}{b^2}\left(\frac{b^3}{3}\ln\frac{b^2}{b^2-y^2} - \frac{y^3}{3}\ln\frac{b+y}{b-y} + \tfrac{2}{3}by^2\right) + \frac{l}{b^4}\left(\frac{b^5}{5}\ln\frac{b^2}{b^2-y^2} - \frac{y^5}{5}\ln\frac{b+y}{b-y} + \tfrac{2}{15}b^3y^2 + \tfrac{2}{5}by^4\right)\right\}.$$

[1]) Nous corrigeons ici une erreur de calcul, qui n'avait aucune influence sur les résultats du chapître premier.

E_s représente le module d'élasticité du schiste.

La forme que prend la surface inférieure du pilier de charbon comprimé par les mêmes pressions sera

$$w_c = \frac{p_0 + k\frac{y^2}{b^2} + l\frac{y^4}{b^4}}{E_c} a.$$

E_c représente le module d'élasticité du charbon.

La partie constante de l'intégral, l'impression au centre ne nous intéresse pas. Il ne s'agit que de la diminution de l'affaissement du dessous du pilier vers les bords.

Pour trouver les coëfficients k et l on mettra

$$\frac{2(m^2-1)}{\pi m^2 E_s}\left\{p_0\left(b\,ln\,\frac{b^2}{b^2-y^2} - y\,ln\,\frac{b+y}{b-y}\right) + \right.$$

$$+\frac{k}{b^2}\left(\frac{b^2}{3}\,ln\,\frac{b^2}{b^2-y^2} - \frac{y^3}{3}\,ln\,\frac{b+y}{b-y} + \tfrac{2}{3}\,b\,y^2\right)$$

$$\left.+\frac{l}{b^4}\left(\frac{b^5}{5}\,ln\,\frac{b^2}{b^2-y^2} - \frac{y^5}{5}\,ln\,\frac{b+y}{b-y} + \tfrac{2}{15}\,b^3\,y^2 + \tfrac{2}{5}\,b\,y^4\right)\right\} = \frac{-k\frac{y^2}{b^2} - l\frac{y^4}{b^4}}{E_c} a.$$

Nous devons calculer

$$p = p_0 + k\frac{y^2}{b^2} + l\frac{y^4}{b^4}.$$

k et l étant proportionnel à p_0.

On voit que seulement

$$\frac{2(m^2-1)}{\pi m^2}\,\frac{E_c}{E_s}\,\frac{b}{a}$$

joue un rôle et qu'une autre proportion de E_c et E_s équivaut à un changement égal dans la proportion b et a.

Avant de continuer nous devons choisir les constantes. Prenons $E_{schiste} = 360.000$ kg/cm² $E_{charbon} = 54.500$ kg/cm². Il nous suffit de prendre $E_s : E_c = 6.6$. Pour le schiste m est très grand. Prenons $m = 10$. Alors $\frac{2(m^2-1)}{\pi m^2}\,\frac{E_c}{E_s} = 0.0955$.

Il est impossible de faire coïncider les deux surfaces sur toute la largeur du pilier de charbon. En effet le relèvement tout près des bords a un caractère différent pour les deux courbes. Mais en prenant k et l égale on obtient une coïncidence quasi parfaite sur une très grande étendue. C'est ce que nous avons représenté dans les figures 12, 13 et 14.

Nous ne prétendons pas contribuer dans cette étude à la théorie de l'élasticité, ce que nous cherchons ce sont les lois physiques pour l'équilibre du terrain houllier autour des exploitations profondes où les pressions surpassent la résistance des matières à la compression.

Les modules d'élasticité du schiste et du grès sont très élevés comparés à celui du charbon. C'est déjà un résultat qui mérite l'attention que notre calcul a démontré que pour les piliers de charbon laissés dans une couche puissante de charbon, c.à d. pour la méthode américaine d'exploitation on peut avec une approximation suffisante considérer le charbon comme de la matière élastique comprimée entre surfaces parallèles et indéformables.

Il est utile d'appliquer pour ce cas la solution donnée par KNEIN [1]), où le frottement est dûment mis en compte. Alors on trouve, que sur une étendue très restreinte dans les coins au ras du front, les pressions sont infiniment grandes, mais en comparant le calcul avec les résultats d'essai on s'aperçoit que ce résultat de calcul n'a pas de signification réelle et qu'on contraire le frottement provoque que le pilier porte plus que la masse de charbon fendue en colonnes mise à la base de nos calculs.

Nous avons donc prouvé que pour les exploitations à faible profondeur ou dans le houllier de forte résistance, le charbon n'est pas morcelé. Dans cette condition il faut l'abattre par machines haveuses et explosifs, tandis que nous traitons à présent les exploitations à pressions tellement élevées que le front du charbon se déplace vers le vide et fait apparaître des plans de clivage soit macro- ou du moins microscopiques.

Annexe 3. Se référant au § 5.

Distribution de la pression sur les couches de charbon. Celui-ci considéré comme de la matière parfaitement plastique.

Nous nous basons pour résoudre ce problème sur un exemple devenu classique, traité dans tous les manuels sur la théorie de la plasticité, celui de la matière plastique pressée entre deux surfaces rugueuses et parallèles. Par ces traités nous savons qu'on peut considérer le frottement selon les surfaces comme constant et égal à p_s la résistance au cisaillement de la matière.

PRANDTL, Zeitschr. Ang. Mech. 1923, p. 401.
HÜTTE, Des Ingenieurs Taschenbuch, I, 1936, p. 347 e).
Quetschen einer plastischen Masse zwischen reibenden Platten.
NÁDAI, Plasticity 1931, p. 221.
Handbuch der Physik, Band VI, Mechanik der Elastischen Körper, 1928, p. 473.

Sur la distance que la matière est exprimée, sur la largeur $b—c$ que le charbon est en mouvement plastique, la pression contre le toit et le mur

[1]) MAX KNEIN, Zur Theorie des Druckversuchs. Abhandlungen aus dem Aerodynamischen Institut an der Technischen Hochschule Aachen. Heft 7, 1927, p. 43—62.

s'accroit linéairement de zéro jusqu'à la pression p_0 du terrain vierge.

Nous trouvons la solution en superposant la compression $-p_0$ dans tous les sens, le chargement de l'hémi-espace représenté dans la partie

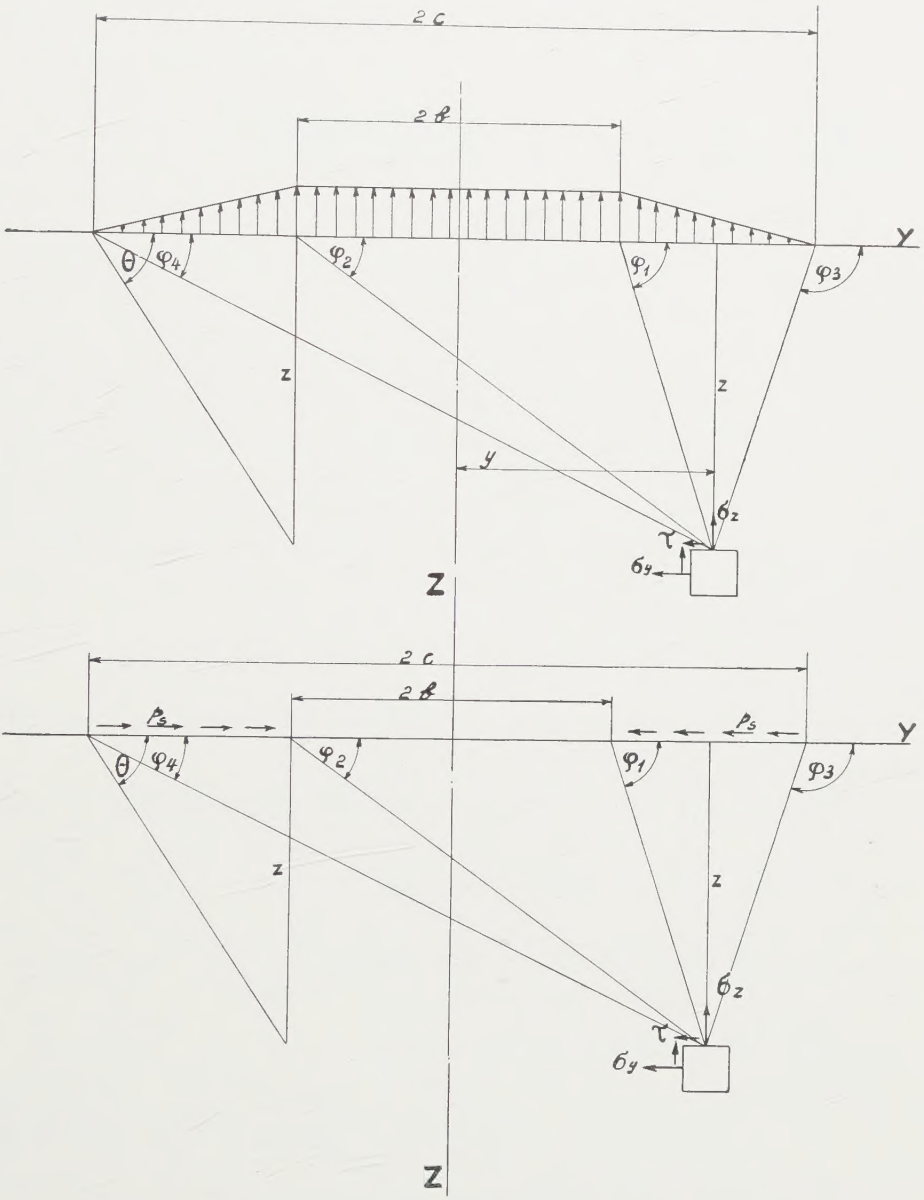

Fig. 20. Notations pour le traitement du problème d'une matière plastique comprimé entre deux hémisphères de matière élastique.

supérieure de la figure 20 et le frottement indiqué dans la partie inférieure de la même figure. A la distance c du centre il règne la pression $-p_0$ dans la masse plastique sur toute la hauteur de la couche d'épaisseur $2a$.

Cette pression tend à pousser la partie plastique de la couche et est en équilibre avec la résistance exercée par les deux surfaces.

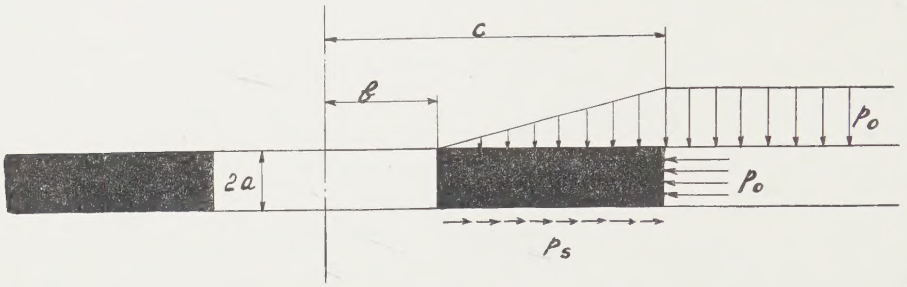

Fig. 21. Equilibre entre la poussée sur la zône plastique et le frottement avec le rocher.

$$p_0 \times 2a = 2p_s(c-b) \quad c-b = \frac{p_0}{p_s}a.$$

Ainsi la distance c jusqu'à laquelle la matière se trouve en état plastique peut être calculée et alors on connait pour chaque point les angles indiqués dans la distribution de la pression verticale dans les figures 20 et 21. Par des intégrations analogues à celles du paragraphe précédant on trouve pour les trois tensions en chaque point y, z

$$\sigma_z = -p_0 + \frac{p_0}{2\pi}(2\varphi_1 - 2\varphi_2 - \sin 2\varphi_1 + \sin 2\varphi_2)$$

$$+ \frac{p_0}{2\pi}\frac{c-y}{c-b}(2\varphi_3 - 2\varphi_1 + \sin 2\varphi_1 - \sin 2\varphi_3) - \frac{p_0}{\pi} tg\,\theta\,(\sin^2\varphi_3 - \sin^2\varphi_1)$$

$$+ \frac{p_0}{2\pi}\frac{c+y}{c-b}(2\varphi_2 - 2\varphi_4 - \sin 2\varphi_2 + \sin 2\varphi_4) - \frac{p_0}{\pi} tg\,\theta\,(\sin^2\varphi_2 - \sin^2\varphi_4)$$

$$+ \frac{p_s}{\pi}(\sin^2\varphi_3 - \sin^2\varphi_1 - \sin^2\varphi_2 + \sin^2\varphi_4)$$

$$\sigma_y = -p_0 + \frac{p_0}{2\pi}(2\varphi_1 - 2\varphi_2 + \sin 2\varphi_1 - \sin 2\varphi_2)$$

$$+ \frac{p_0}{2\pi}\frac{c-y}{c-b}(2\varphi_3 - 2\varphi_1 + \sin 2\varphi_3 - \sin 2\varphi_1) +$$

$$+ \frac{p_0}{\pi} tg\,\theta\left(ln\frac{\sin^2\varphi_3}{\sin^2\varphi_1} - \sin^2\varphi_3 + \sin^2\varphi_1\right)$$

$$+ \frac{p_0}{2\pi}\frac{c+y}{c-b}(2\varphi_2 - 2\varphi_4 + \sin 2\varphi_2 - \sin 2\varphi_4) +$$

$$+ \frac{p_0}{\pi} tg\,\theta\left(ln\frac{\sin^2\varphi_4}{\sin^2\varphi_2} + \sin^2\varphi_2 - \sin^2\varphi_4\right)$$

$$+ \frac{p_s}{\pi}\left(ln\frac{\sin^2\varphi_3\sin^2\varphi_4}{\sin^2\varphi_1\sin^2\varphi_2} - \sin^2\varphi_3 + \sin^2\varphi_1 + \sin^2\varphi_2 - \sin^2\varphi_4\right)$$

$$\tau = \frac{p_0}{2\pi}(\cos 2\varphi_1 - \cos 2\varphi_2)$$

$$+ \frac{p_0}{2\pi} tg\,\theta\,(\sin 2\varphi_3 - \sin 2\varphi_1 + 2\varphi_3 - 2\varphi_1) - \frac{p_0}{\pi}\frac{y-c}{c-b}(\sin^2\varphi_3 - \sin^2\varphi_1)$$

$$- \frac{p_0}{2\pi} tg\,\theta\,(\sin 2\varphi_2 - \sin^2\varphi_4 + 2\varphi_3 - 2\varphi_4) + \frac{p_0}{\pi}\frac{c+b}{c-b}(\sin^2\varphi_2 - \sin^2\varphi_4)$$

$$+ \frac{p_s}{2\pi}(2\varphi_3 - 2\varphi_1 - 2\varphi_2 + 2\varphi_4 + \sin 2\varphi_3 - \sin 2\varphi_1 - \sin 2\varphi_2 + \sin 2\varphi_4).$$

Avec ces formules nous avons construit la figure 16, où quelques absurdités ou impossibilités sautent aux yeux.

Pour bien comprendre l'applicabilité de la théorie de la plasticité et pour mieux apprécier la valeur des calculs suivants il faut que nous nous étendions sur les anomalies aux points $z = 0$, $y = b$ et $y = c$.

Comme presque toujours, les équations différentielles simultanées, qui doivent conduire à la répartition des tensions dans la masse plastique et aux lignes de glissement, ne permettent pas une solution générale et il faut choisir une répartition simple, une distribution spéciale des tensions au contour de la masse qui donne une solution rationelle.

La solution connue, qui a servi de base à nos calculs est celle correspondante à la distribution des tensions au contour représentée en la figure 22.

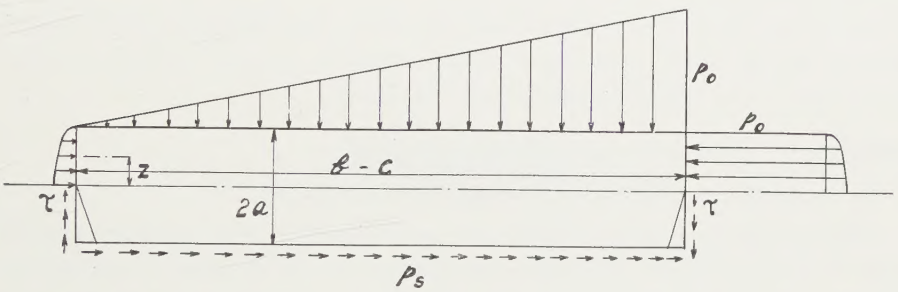

Fig. 22. Distribution des pressions et des tensions de cisaillement, autour de la masse plastique, un peu idéalisée (d'après PRANDTL—NÁDAI—HÜTTE) pour permettre la solution des équations d'équilibre.

En vérité le front du charbon est exempt de pression et de tensions de cisaillement, mais nous adoptons sur le front nu du charbon la pression

$$\sigma_y = 2\,p_s\sqrt{1 - \frac{y^2}{a^2}}.$$

C'est une pression très légère distribuée selon une ellipse et aussi la tension de cisaillement

$$\tau = p_s\frac{z}{a}.$$

C'est une force de cisaillement nulle et des tensions insignifiantes.

Sur la surface de la masse plastique située à l'intérieur de la couche nous calculons avec la pression

$$\sigma_y = p_0 + 2\, p_s \sqrt{1 - \frac{z^2}{a^2}}$$

au lieu de p_0 seulement. Ainsi nous n'ajoutons aucune force résultante sur la masse plastique. Les tensions de cisaillement ajoutées sont

$$\tau = -p_s \frac{z}{a}$$

mais ne donnent non plus de forces de cisaillement sur toute l'épaisseur de la masse plastique.

Les tensions τ devaient être ajoutées pour donner l'équilibre des tensions jusque dans les coins même. En vérité elles n'existent pas sur la surface libre et alors le frottement disparait dans le coin même sur une très petite étendue. Les anomalies représentées dans la figure 12 peuvent être rayées. Mais la solution exacte tout près des coins n'est pas connue.

La solution est parfaitement acceptable à quelque distance du front $y = b$ et de la limite de plasticité, chez $y = c$. Pour ces limites elle se simplifie en

$$\tau = 0 \text{ et } \sigma_y = p_0 \qquad \sigma_z = p_0 \qquad \tau = 0.$$

Mais alors nous pouvons simplifier cette sorte de problèmes en admettant que σ_x est constante dans toute l'épaisseur de la couche supposée comme assez étendue en largeur, comparée à l'épaisseur.

Annexe 4. Se référant au § 6.

La pression sur une couche de matière incohérente.

Nous avons exposé dans une étude sur le fonçage des puits de mine [1]) que la condition pour le glissement à l'intérieur d'un massif pulvérulent est que la pression agissante sur une surface de glissement devient tangente au cercle de Mohr. L'angle entre cette pression et la surface de glissement est égale à l'angle du talus naturel. On trouve ce théorème expliqué dans plusieurs traités [2]).

Pour le charbon tout venant le talus est de 45°. Et pour calculer un exemple en chiffres nous prenons cet angle.

Dans la figure 23 nous avons représenté le cercle de Mohr pour ce cas. La figure 23 nous apprend

$$\varrho_1 = \varrho_2 \frac{1 + \sin \varphi}{1 - \sin \varphi}.$$

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[1]) De Ingenieur 1930, No. 4, M. p. 20 et No. 52, M. p. 31.

[2]) Les Cuvelages par L. DENOËL 1935, p. 46.
Equilibre des massifs à frottement interne par ALB. CAQUOT 1934.

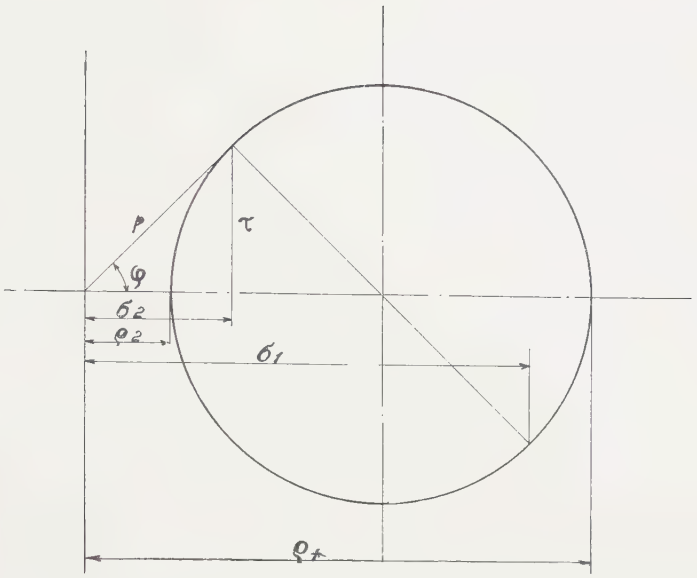

Fig. 23. Cercle de MOHR représentant la condition de glissement dans un massif pulvérulent. Dans chaque point de la zone plastique la tension p est tangente du cercle et fait l'angle φ avec la surface où il agit. $\varphi =$ angle du talus naturel, σ composante normale de p, τ composante tangentielle en même temps résistance au glissement interne. ϱ_1 et ϱ_2 tensions principales au même point.

Pour $\varphi = 45°$ $\quad \varrho_1 = 5{,}8\ \varrho_2$

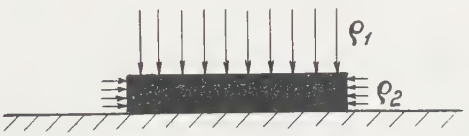

Fig. 24. Contraintes d'un élément dans un massif incohérent au moment de glissement.

Quelle pression ϱ_1 peut supporter un élément d'une matière pulvérulente, appuyée latéralement par la pression ϱ_2. La réponse est donnée par la condition de glissement

$$\varrho_1 = \varrho_2 \frac{1 + \sin \varphi}{1 - \sin \varphi}.$$

En général le schiste du toit et du mur est luisant et le coëfficient de frottement est petit. Nous le dénommons f et écrivons l'équilibre pour un élément de hauteur $2a$ et de largeur dy quand il se déplace vers le vide

$$2a \times dp_y = 2 f p_z \, dy \qquad p_z = \frac{1 + \sin \varphi}{1 - \sin \varphi} p_y$$

$$\frac{dp_y}{p_y} = f \frac{1 + \sin \varphi}{1 - \sin \varphi} d \frac{y}{a}.$$

Après intégration

$$ln \frac{p_y}{p_0} = f \frac{1 + sin\,\varphi}{1 - sin\,\varphi} \frac{y}{a}$$

$$p_y = p_0\, e^{f \frac{1+sin\varphi}{1-sin\varphi} \frac{y}{a}} \qquad p_z = \frac{1 + sin\,\varphi}{1 - sin\,\varphi}\, p_0\, e^{f \frac{1+sin\varphi}{1-sin\varphi} \frac{y}{a}}.$$

Avec la formule

$$y = a \frac{1 - sin\,\varphi}{1 + sin\,\varphi} \frac{1}{f} ln \frac{1 - sin\,\varphi}{1 + sin\,\varphi} \frac{p_z}{p_0}$$

on peut calculer à quelle distance y atteint la pression correspondante à la profondeur

$$p = \frac{h\,\gamma}{10} \text{ kg par cm}^2.$$

Le résultat est qu'avec un peu d'appui, un peu de boisage une matière incohérente peut supporter des pressions très élevées tout près du front. C'est ce que dit la loi exponentielle.

Pour terminer nous calculons encore la répartition de la pression près du front du charbon pour la condition d'écrasement du charbon qui se rapproche de la réalité.

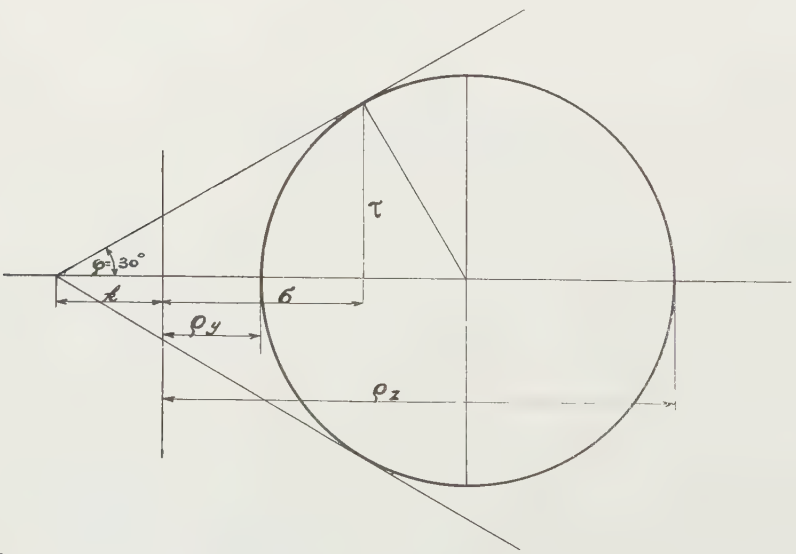

Fig. 25. Condition de la résistance au glissement à l'intérieur d'une couche de charbon. Les confins des cercles de MOHR sont deux lignes à 30° avec l'axe.

Le charbon peut résister à une petite tension et des essais de résistance à la pression sur éprouvettes soumises à pression dans tous les sens ont démontré que la représentation de la condition de rupture est assez bien

exprimée par la figure 25. Les cercles de Mohr touchent aux deux tangentes sous 30° avec l'absise.

La condition de glissement dans le massif du charbon selon la figure est

$$\sin \varphi = \frac{\frac{\varrho_z - \varrho_y}{2}}{\frac{\varrho_z + \varrho_y}{2} + k}$$

et on obtient la formule

$$\varrho_2 = \frac{2 \sin \varphi}{1 - \sin \varphi} \, k \, e^{f \frac{1+\sin\varphi}{1-\sin\varphi} \frac{y}{a}}$$

que nous représentons dans la figure 26 pour $f = 1/10$.

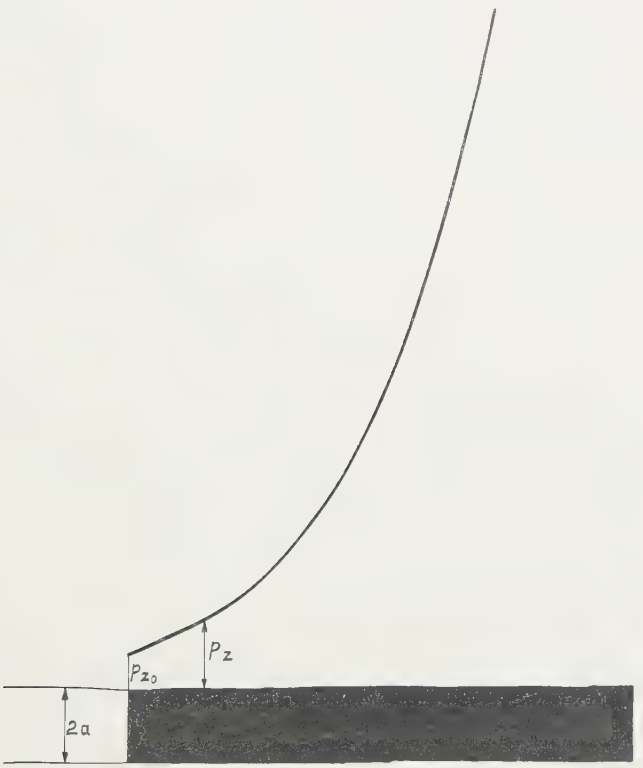

Fig. 26. Augmentation rapide de la pression sur le charbon devant le front selon une loi exponentielle, dessinée à échelle. p_{z_0} est la résistance à l'écrasement du charbon.

Nous devons observer que nous appelons la pression du toit sur le charbon une pression principale ϱ_z. Mais il faut observer qu'elle est accompagnée d'un peu de frottement, d'une tension de cisaillement $f\varrho_z$. L'approximation est absolument permise pour les applications techniques.

Du reste il est possible d'appliquer des formules analogues à celles utilisées dans l'annexe 3 [1]).

Si l'on poursuit les calculs avec les équations données par HARTMANN on trouve exactement la même répartition de la pression p_z selon les surfaces de contact et on peut indiquer que pour les parties près du front et près de la limite qui sépare la partie écrasée de la partie déformée élastiquement, la théorie complète doit se contenter aussi d'une acceptation arbitraire mais peu importante.

Il est à noter que même dans le charbon friable la pression peut accroître à des valeurs excessives, surpassant sans limites la résistance à la rupture du toit et du mur, à très courte distance y grace au frottement et à la loi exponentielle.

Les calculs ont été controlés par M. A. HELLEMANS, ingénieur physicien et électricien, ingénieur des travaux de fond à la mine Hendrik. M. HELLEMANS, ainsi que M. G. E. TUMMERS, ingénieur physicien des laboratoires des recherches mécaniques de la mine Emma ont bien voulu discuter avec moi les problèmes traités dans cette étude.

[1]) Handbuch der Physik, Band VI. Das Gleichgewicht lockerer Massen, p. 493.

Hydrodynamics. — *On the application of viscosity data to the determination of the shape of protein molecules in solution.* *) By J. M. BURGERS. (Mededeeling N°. 38 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hoogeschool te Delft.)

(Communicated at the meeting of March 30, 1940.)

Formulae for some model systems consisting of rigidly connected spheres of equal radius.

6. *Model system consisting of two rigidly connected spheres.* — In the preceding sections the results of experimental investigations on the effective viscosity of suspensions of certain proteins and on the sedimentation velocities of these proteins have been compared with the results derived from theoretical calculations, based upon the assumption that the protein molecules might be treated as rotational ellipsoids, either elongated or flattened. It was found that there always remained discrepancies between calculated and observed results, in the sense that the sedimentation velocities came out too low, when the dimensions were calculated with the aid of the observed values of η_{sp}.

Now ellipsoidal bodies, whether elongated or oblate, have a rather large concentration of mass near the centre. It can be supposed that with bodies of other types, where the mass is more dispersed towards the ends, the specific increase of the viscosity will be enhanced relatively to the frictional constant. It is possible that such an effect will tend to diminish the discrepancy which was found when making the calculations for ellipsoids.

An approximate calculation can be made without great difficulty for an extreme case, in which the molecule is assumed to consist of *two spheres* of equal radius R, joined by a rigid link (which itself is assumed not to influence the motion of the liquid and which, therefore, will be left out of consideration), so that the centres of the spheres are at the constant distance a from each other (see fig. 1a). Such a system has already been

*) Continued from these Proceedings **43**, 315 (1940). — Dr. K. O. PEDERSEN in a letter to the author has pointed out that the value 150000 for the molecular weight of serum globulin, mentioned in the preceding part in Table I and used in the calculations concerning the sedimentation velocity, must be considered as too low, and that the value 167000, given in Table 5, p. 44, of "The Ultracentrifuge", is assumed to be the most probable one. He further mentions that the behaviour of gliadin shows that the molecules of this particular protein are thread-shaped. — In connection with footnote 5) the following paper should be mentioned: A. POLSON, Untersuchungen über die Diffusionskonstanten der Proteine, Kolloid-Zeitschr. **87**, 149—181 (1939).

considered by KUHN [23]). A discussion of his work is also given by ROBINSON [24]). These authors, however, the same as HUGGINS, who has given an extension of KUHN's method [25]), all have based the calculations of the forces exerted by the liquid on the spheres upon STOKES' formula for the resistance of a sphere, without giving attention to the influence of the distortion of the field around one sphere upon the flow along the other. Although a rigorous treatment of a system consisting of several spheres would be difficult [26]), an approximation can be obtained in the following relatively simple way, and it may be worth while to consider not only the system consisting of two spheres, but also some other cases.

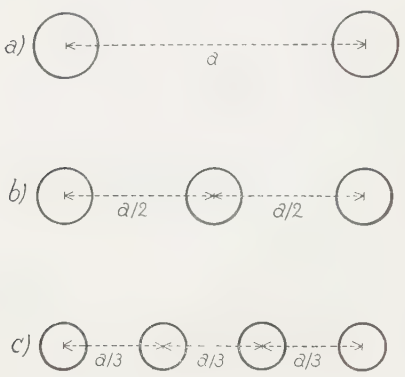

Fig. 1a—c. Systems consisting of 2, 3 and 4 spheres in a straight line.

7. In order to calculate the *resistance* experienced by the system of two spheres, in the case of *longitudinal* motion, we replace each sphere by a force F, acting at its centre in the direction of the motion. Introducing coordinates x, y, z with the origin midway between the two spheres, the x-axis along the axis of the system, and the y- and z-axes perpendicular to it, the x-component of the velocity at an arbitrary point of the field is given by [27]):

$$u = \frac{F}{8\pi\eta}\left\{\frac{1}{r_1} + \frac{(x+a/2)^2}{r_1^3}\right\} + \frac{F}{8\pi\eta}\left\{\frac{1}{r_2} + \frac{(x-a/2)^2}{r_2^3}\right\} \quad . \quad . \quad (11)$$

where r_1 is the distance of the point considered from the centre of the sphere at $x = -a/2$, and r_2 the distance from the centre of the other sphere. We calculate the mean value of this expression over the surface of, say, the sphere with its centre at $x = -a/2$, and require that this mean value shall be equal to the velocity U_{long} of the system. Assuming that the ratio R/a is a small quantity and neglecting quantities of the order R^3/a^3, we obtain:

$$U_{long} = \frac{F}{8\pi\eta}\frac{4}{3R} + \frac{F}{8\pi\eta}\frac{2}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

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[23]) W. KUHN, Zeitschr. f. physik. Chemie A **161**, 3 (1932).

[24]) J. R. ROBINSON, Proc. Roy. Soc. (London) A **170**, 528 (1939).

[25]) M. L. HUGGINS, Journ. of Phys. Chemistry **42**, 911 (1938); **43**, 439 (1939).

[26]) The hydrodynamical problem of two spheres has been treated by H. FAXÉN and also by Miss M. STIMSON and G. B. JEFFERY; compare C. W. OSEEN, Hydrodynamik, pp. 160—162. — Our approximation substantially corresponds to the treatment given by OSEEN at pp. 157—160 as a "first approximation".

[27]) See the equations given in the "Second Report", p. 119, eqs. (4.1) and (4.2).

In the case of *transverse* motion, e.g. in the direction of the y-axis, the forces F likewise must be in the direction of this axis, and the equation for the y-component of the velocity becomes:

$$v = \frac{F}{8\pi\eta}\left(\frac{1}{r_1} + \frac{y^2}{r_1^3}\right) + \frac{F}{8\pi\eta}\left(\frac{1}{r_2} + \frac{y^2}{r_2^3}\right) \quad . \quad . \quad . \quad . \quad (13)$$

We require that the mean value of this expression over the surface of a sphere shall be equal to the velocity U_{trans} of the system. With the same degree of approximation we obtain:

$$U_{trans} = \frac{F}{8\pi\eta}\,\frac{4}{3R} + \frac{F}{8\pi\eta}\,\frac{1}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

From these equations the frictional constants of the system can be found, and the mean frictional constant for all directions in space can be calculated. It is convenient to bring into evidence the radius R of the spheres, and to write the formula for f_m as follows:

$$f_m = 6\pi\eta R N_A/\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

in which case the formula for the sedimentation constant becomes:

$$S = \frac{1 - \varrho V}{6\pi\eta}\,\frac{M}{N_A R}\,\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

After a short calculation it is found that the factor λ occurring in these formulae is given by:

$$\lambda = \frac{1}{2}\left(1 + \frac{R}{a}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

It will be seen that the reciprocal influence of the two spheres upon each other tends to diminish the resistance of the system (this effect is larger in the case of the longitudinal motion than in the case of the transverse motion).

8. In order to find the influence of the system upon the *effective viscosity* of a liquid in shearing motion, we follow the treatment given in the "Second Report", pp. 132—137. According to eq. (10. 4c), *loc. cit.*, at the centre of each sphere there exists an axial velocity component, directed outwards, of the magnitude:

$$U_{ax} = \frac{1}{2}\,\varkappa\, a \sin^2\theta \sin\Phi \cos\Phi \quad . \quad . \quad . \quad . \quad . \quad (18)$$

In order to annul this velocity over the surface of each sphere as well as possible we again introduce forces at the centres of the spheres, which

forces now both will be directed inwards, and consequently to a certain extent will counteract each others effect. By a similar reasoning as was used to derive eq. (12) above we obtain the following condition, which ensures that the mean value of the axial component of the flow set up by these forces (which, as they are directed inwards, will be denoted by $-F$) shall balance the velocity U_{ax}:

$$U_{ax} = -\frac{F}{8\pi\eta}\frac{4}{3R} + \frac{F}{8\pi\eta}\frac{2}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

from which:

$$F = -\frac{6\pi\eta R\, U_{ax}}{1-3R/2a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

As the strength of the doublet formed by these forces has the value $M = aF$, the contribution to η_{sp} due to a particle held in a definite position is given by (comp. "Second Report", p. 134, eq. (9. 8) and p. 137):

$$-\frac{M}{\eta\varkappa}\sin^2\theta\sin\phi\cos\phi = +\frac{3\pi a^2 R}{1-3R/2a}\sin^4\theta\sin^2\phi\cos^2\phi.$$

Assuming that all positions in space are equally probable for the axis of the system, the mean value of the goniometrical expression in this formula becomes 1/15, and the average contribution to η_{sp} can be written:

$$\frac{8\pi R^3}{3}\cdot\frac{3}{40}\frac{a^2/R^2}{1-3R/2a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where the factor $8\pi R^3/3$ represents the volume of the system [28]).

To this amount must be added the contribution which each sphere already gives in consequence of its local field, independently of the force F introduced at its centre [29]). According to EINSTEIN's formula this con-

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[28]) The same result can be obtained by calculating the dissipation, which is the method followed by KUHN and by HUGGINS (*l.c.* footnotes 23) and 25) above). The dissipation in unit time connected with the forces F is given by:

$$-2FU_{ax} = \frac{3\pi\eta\varkappa^2 a^2 R}{1-3R/2a}\sin^4\theta\sin^2\phi\cos^2\phi.$$

When this expression is divided by $\eta\varkappa^2$ and the mean value is taken for all directions in space, we fall back upon (21).

[29]) That in systems consisting of spheres at relatively large distances from each other this contribution should be added, is remarked by W. KUHN, Zeitschr. f. physik. Chemie A **161**, 22 (1932). In the case of systems where the constituent parts fit more closely together the amount to be added probably will be less. It will decrease to zero for cylindrical systems of the type considered in the "Second Report". See M. L. HUGGINS, Journ. of Phys. Chemistry **43**, 442 (1939).

tribution for the two spheres together is given by: $8\pi R^3/3 \,.\, 2{,}5$; hence we find that the coefficient A_{II} in formula (1) for η_{sp} will take the value:

$$A_{II} = \frac{3}{40} \frac{a^2/R^2}{1 - 3\,R/2a} + 2{,}5 \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Neglecting terms of the order R^3/a^3 this result also may be written:

$$A_{II} = \frac{3}{40} \frac{a^2}{R^2} + \frac{9}{80} \frac{a}{R} + 2{,}67 \quad . \quad . \quad . \quad . \quad . \quad . \quad (22a)$$

9. The formulae derived in the preceding sections must be considered as approximations, which can be used only for sufficiently large values of the ratio a/R. Consequently from the cases considered in Table I we shall pick out those proteins only, for which the values of η_{sp}/cV are relatively large. The following results then are obtained (the radius R of the spheres is found from the volume of the molecule, already given in Table I; the ratio a/R is calculated from the value of η_{sp}/cV; with the aid of a/R the value of λ is found):

Name of the protein	$10^8 R$	a/R	$10^8 a$	λ	$10^{13} . S_{calc.}$	$10^{13} . S_{obs.}$
Gliadin	15.7	11.9	187	0.542	2.27	2.1
Serum globulin *)	29.1	8.47	246	0.559	7.15	7.1
Thyroglobulin	45.2	9.07	410	0.555	19.5	19.2
Octopus haemocyanin	74.2	8.48	630	0.559	47.8	49.3

*) Molecular weight taken as 167000.

Hence with this model we arrive at values for the sedimentation constant which are quite near to the observed values.

The result, of course, does not pretend to be more than an exemplification of what can be obtained when forms of molecules are considered, differing from the exact ellipsoidal type. Before any conclusions can be drawn it will be necessary to consider the other models.

10. *Linear systems consisting of three and four spheres.* — Two other cases which can be worked out almost equally simply are those indicated resp. in fig. 1*b* (three equal spheres in one line, at distances $a/2$) and fig. 1*c* (four equal spheres in one line, at distances $a/3$). The forces acting on the central sphere (or spheres) of course will be different from those acting on the outer spheres, and it is necessary to write down the equations separately for an inner sphere and for an outer sphere. The following results are obtained (in all cases a is the distance between the centres of the outer spheres, and R the radius of a single sphere):

system of 3 spheres in one line:

$$\lambda = \frac{1}{3}\left(1 + \frac{10}{3}\frac{R}{a} - \frac{1}{4}\frac{R^2}{a^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

$$A_{II} = \frac{1}{20}\frac{a^2}{R^2} + \frac{3}{40}\frac{a}{R} + 2{,}61 \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

system of 4 spheres in one line:

$$\lambda = \frac{1}{4}\left(1 + \frac{13}{2}\frac{R}{a} - \frac{9}{8}\frac{R^2}{a^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

$$A_{II} = \frac{1}{24}\frac{a^2}{R^2} + \frac{3}{160}\frac{a}{R} + 2{,}54 \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

When these formulae are applied to the proteins considered above it is found that the sedimentation velocity decreases when we pass from two spheres to three and to four spheres. This is a consequence of the circumstance that the force acting on a single sphere decreases more rapidly than does the radius (and thus the resistance) of the sphere. The existing mutual influence of the spheres to a certain extent counteracts this decrease, but not sufficiently to prevent the decrease of S.

The numerical results will be given afterwards.

11. *System of 8 spheres in the corners of a cube.* — Two further cases which can be worked out, are those of four spheres, lying at the corners of a square, and of eight spheres, lying at the corners of a cube. In these cases somewhat more complicated systems of forces are required. We shall briefly indicate the main points of importance which occur in the calculation of the *cubical* system.

As the equations for the field of flow are linear, the *resistance* can be calculated separately for the motion in the direction of each of the three coordinate axes. In the case of the cubical system, moreover, symmetry makes it sufficient to consider the motion in the direction of one axis only, and further helps us in fixing the general character of the force system, which has been indicated schematically in fig. 2. By expressing the condition that the mean velocity over the surface of each one of the spheres must have the value U, and that there shall be no transverse velocity, two equations are obtained, which are sufficient to determine both f and g.

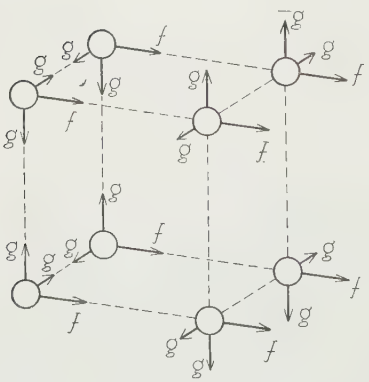

Fig. 2. System of forces in the case of rectilinear motion in the direction of one of the coordinate axes.

We pass over the details of the calculations and mention only the following results: $g/f = -0{,}409\ R/a - 0{,}64\ R^2/a^2$, and

$$U = \frac{f}{6\pi\eta R}\left(1 + 5{,}70\,\frac{R}{a} - 0{,}34\,\frac{R^2}{a^2}\right) \quad . \quad . \quad . \quad . \quad (27)$$

As $f = W/8$, the factor λ occurring in eqs. (15) and (16) will have the value:

$$\lambda = \frac{1}{8}\left(1 + 5{,}70\,\frac{R}{a} - 0{,}34\,\frac{R^2}{a^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Here a has been written for the length of a side of the cube.

12. In order to calculate the contribution of the system to the effective viscosity of a liquid in shearing motion, it is necessary first to consider a given position in space of the system, which can be defined by a set of Eulerian angles. This has been indicated in fig. 3, where $O\,x\,y\,z$ is a system

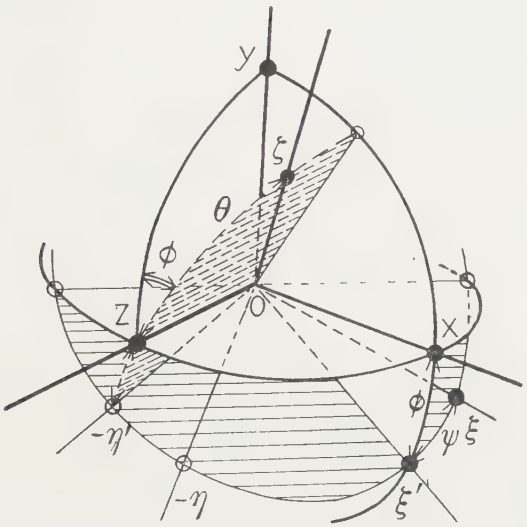

Fig. 3. Relative position in space of the systems of coordinates $O\,x\,y\,z$ (connected with the motion of the liquid) and $O\,\xi\,\eta\,\zeta$ (rigidly connected with the cubical system or with the square system).

of coordinates, fixed in space, with reference to which the shearing motion of the liquid is defined by means of the equation:

$$U = \varkappa y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

The ζ-axis of the system $O\,\xi\,\eta\,\zeta$, which is rigidly connected with the cube, has the direction determined by the angles Φ ($= \angle\, y\,z\,\zeta$) and θ ($= \angle\, z\,O\,\zeta$). The ξ, η-plane, perpendicular to $O\,\zeta$, cuts the plane $y\,O\,x$ along the line $O\,\xi'$, which will be taken as the ξ'-axis of an auxiliary system $O\,\xi'\,\eta'\,\zeta$; it will be seen that the angle $x\,O\,\xi'$ is equal to Φ. Finally the ξ-axis

is derived from the ξ'-axis by a rotation through the angle ψ ($= \angle \xi' O \xi$).

When for shortness we write x_1, x_2, x_3 instead of x, y, z, and ξ_1, ξ_2, ξ_3 instead of ξ, η, ζ, the relation between the two sets of coordinates can be expressed by the formulae:

$$\xi_i = \sum_j a_{ij}\, x_j, \qquad x_i = \sum_j a_{ji}\, \xi_j \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where $a_{ij} = \cos(\xi_i, x_j)$, which quantities easily can be expressed by means of the sines and cosines of the angles Φ, θ, ψ.

The shearing motion of the liquid, given by (29), when described with respect to the new system, becomes (writing $\dot{\xi}_i$ for $d\xi_i/dt$):

$$\dot{\xi}_i = \varkappa \sum_j a_{i1}\, a_{j2}\, \xi_j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

By means of this formula the components of the velocity of the liquid at the centres of the 8 spheres can be calculated; from these components the rotational velocity which is imparted to the system can be derived; finally we can obtain the components of the remaining relative velocity of the liquid with respect to the spheres. In the present case the rotational velocities are determined by the formulae [30]):

$$(\dot{\xi}_i)_{rot} = \sum_j \omega_{ij}\, \xi_j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32a)$$

and the remaining relative velocities by:

$$(\dot{\xi}_i)_{rel} = \sum_j D_{ij}\, \xi_j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32b)$$

where:

$$\omega_{ij} = \tfrac{1}{2} \varkappa (a_{i1}\, a_{j2} - a_{j1}\, a_{i2}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (33a)$$

$$D_{ij} = \tfrac{1}{2} \varkappa (a_{i1}\, a_{j2} + a_{j1}\, a_{i2}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (33b)$$

Now a set of forces must be found, acting at the centres of the spheres and producing a flow, the mean value of which over the surface of each sphere just annuls the remaining relative velocity of the liquid, calculated at the centre of that sphere. On account of the symmetry it is convenient first to treat separately the cases

$$\dot{\xi}_1 = D_{11}\, \xi_1, \qquad \dot{\xi}_2 = \dot{\xi}_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

and:

$$\dot{\xi}_1 = D_{12}\, \xi_2, \qquad \dot{\xi}_2 = D_{12}\, \xi_1, \qquad \dot{\xi}_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad (B)$$

---

[30]) In a more general (*i.e.* less symmetrical) case the analysis into rotation and remaining relative velocities must be made in such a way that the force system which is introduced to annul the remaining relative velocities does not give rise to a resulting moment about the centre of the system of spheres. An example will be given in section **16**. As will be readily understood, the force system neither may have a resultant.

The systems of forces for these two cases have been represented in figs. 4 and 5. In case A the following values are found for f and g:

$$f = -3\pi\eta\, RaD_{11}\, k_1\,; \qquad g = -3\pi\eta\, RaD_{11}\, k_2 \quad . \quad . \quad . \quad (34)$$

where:

$$\left.\begin{aligned} k_1 &= 1 + 1{,}639\, R/a + 3{,}02\, R^2/a^2 \\ k_2 &= \quad 0{,}409\, R/a + 1{,}51\, R^2/a^2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (35)$$

while in case B:

$$f = -3\pi\eta\, RaD_{12}\, l_1\,; \qquad g = -3\pi\eta\, RaD_{12}\, l_2 \quad . \quad . \quad . \quad (36)$$

where:

$$\left.\begin{aligned} l_1 &= 1 + 0{,}018\, R/a + 0{,}03\, R^2/a^2 \\ l_2 &= -0{,}242\, R/a - 0{,}35\, R^2/a^2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (37)$$

13. These forces now must be combined into doublets.

In case A (fig. 4) there are four doublets of strength fa, directed

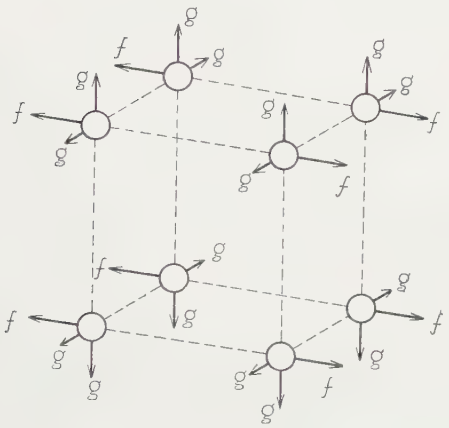

Fig. 4. System of forces called forth by the relative motion of the liquid in case A.

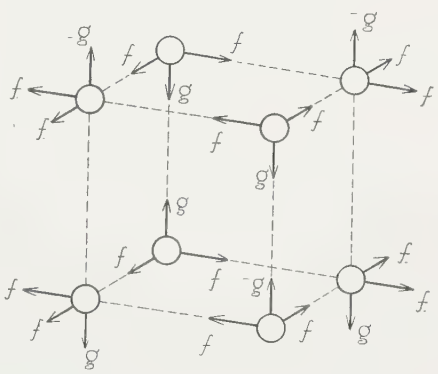

Fig. 5. System of forces called forth by the relative motion of the liquid in case B.

parallel to the ξ_1-axis; four doublets of strength ga parallel to the ξ_2-axis, and four doublets of the same strength ga parallel to the ξ_3-axis. Making use of eq. (9.8), "Second Report", p. 134, the contribution to the expression for η_{sp} derived from these doublets will become:

$$\left.\begin{aligned} &+\frac{12\pi Ra^2}{\varkappa} D_{11}\,(k_1\, a_{11}\, a_{12} + k_2\, a_{21}\, a_{22} + k_2\, a_{31}\, a_{32}) = \\ &= \frac{12\pi Ra^2}{\varkappa^2}\,(k_1\, {D_{11}}^2 + k_2\, D_{11}\, D_{22} + k_2\, D_{11}\, D_{33}) \end{aligned}\right\} \quad . \quad . \quad (38)$$

When again it is assumed that all positions in space are equally probable

for the cube, the mean value of this expression can be calculated. It is found that $\overline{D_{11}^2} = \varkappa^2/15$, $\overline{D_{11}D_{22}} = -\varkappa^2/30$, etc., and the result becomes:

$$12\,\pi\,Ra^2\,(k_1/15 - k_2/15) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

A similar calculation can be made with reference to D_{22} and D_{33}; the same result is obtained, so that in the final expression the amount (39) is multiplied by 3.

Next considering case *B*, it will be seen from fig. 5 that there appear two doublets of strength $f\sqrt{2}\,.\,a\sqrt{2} = 2\,f\,a$, in the direction of the diagonal of the first and third quadrants formed by the ξ_1- and ξ_2-axis; two doublets of strength $-2\,f\,a$ in the direction of the diagonal of the second and third quadrants; and 2 doublets of strength $g\,a$ and 2 doublets of strength $-g\,a$ in the direction of the ξ_3-axis. The latter doublets of course balance each other. As to the others, it is not difficult to deduce from eq. (9. 8), "Second Report", p. 134, that their contribution to η_{sp} is given by:

$$\left.\begin{aligned} &+\frac{12\,\pi\,Ra^2}{\varkappa}\,D_{12}\,l_1\left\{\frac{(a_{11}+a_{21})\,(a_{12}+a_{22})}{2} - \frac{(a_{11}-a_{21})\,(a_{12}-a_{22})}{2}\right\} = \\ &= +\frac{12\,\pi\,Ra^2}{\varkappa}\,l_1\,D_{12}\,(a_{11}\,a_{22} + a_{21}\,a_{12}) = \frac{24\,\pi\,Ra^2}{\varkappa^2}\,l_1\,D_{12}^2 \end{aligned}\right\} . \quad (40)$$

The mean value of this expression, for all spatial directions of the cube, is equal to:

$$24\,\pi\,Ra^2\,l_1/20 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

As contributions of the same magnitude are obtained from D_{13} and D_{23}, the amount (41) likewise must be multiplied by 3.

Hence the resulting expression for the average contribution of the whole system to η_{sp} becomes:

$$12\,\pi\,Ra^2\,(k_1/5 - k_2/5 + 3\,l_1/10) = \frac{32\,\pi\,R^3}{3}\cdot\frac{a^2}{R^2}\left(\tfrac{9}{40}\,k_1 - \tfrac{9}{40}\,k_2 + \tfrac{27}{80}\,l_1\right). \quad (42)$$

Inserting the values of k_1, k_2, l_1 given in (35) and (37), and adding the EINSTEIN term 2,5, we finally obtain:

$$A_{II} = \frac{9}{16}\,\frac{a^2}{R^2} + 0{,}282\,\frac{a}{R} + 2{,}85 \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

14. It may be useful to observe that the same result (42) can be found by calculating the dissipation. The total dissipation in unit time connected

with all the forces f and g, for a definite position of the cube, is given by the following expression:

$$12\pi\eta Ra^2[k_1(D_{11}^2+D_{22}^2+D_{33}^2)+2k_2(D_{11}D_{22}+D_{11}D_{33}+D_{22}D_{33})+ \\ +2l_1(D_{12}^2+D_{13}^2+D_{23}^2)] \quad (44)$$

which, in consequence of the relation $D_{11}+D_{22}+D_{33}=0$, also can be written:

$$12\pi\eta Ra^2[(k_1-k_2)(D_{11}^2+D_{22}^2+D_{33}^2)+2l_1(D_{12}^2+D_{13}^2+D_{23}^2)]. \quad (44a)$$

This expression remains valid also when the motion of the liquid should be of a more general type than is described by eq. (29). Let us assume that with reference to the system x_1, x_2, x_3 the flow is given by:

$$\dot{x}_k = \sum_l \varkappa_{kl} x_l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

We write:

$$\delta_{kl} = \tfrac{1}{2}(\varkappa_{kl}+\varkappa_{lk}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45a)$$

then with reference to the system ξ_1, ξ_2, ξ_3 instead of (33b) we obtain:

$$D_{ij} = \sum_{kl} \alpha_{ik}\alpha_{jl}\delta_{kl} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Evidently we have: $D_{11}+D_{22}+D_{33}=\delta_{11}+\delta_{22}+\delta_{33}=0$. When again all positions in space of the ξ_1, ξ_2, ξ_3-system are considered as equally probable, and mean values are calculated, it is found that:

$$\overline{D_{11}^2+D_{22}^2+D_{33}^2}=\tfrac{1}{5}I \quad ; \quad \overline{D_{12}^2+D_{13}^2+D_{23}^2}=\tfrac{3}{20}I \quad . \quad (47)$$

where

$$I=2(\delta_{11}^2+\delta_{22}^2+\delta_{33}^2)+4(\delta_{12}^2+\delta_{13}^2+\delta_{23}^2) \quad . \quad . \quad . \quad (48)$$

which is the "invariant of the second degree" of the quantities δ_{kl}.

Hence the average dissipation in unit time due to the system of 8 spheres, apart from the EINSTEIN term, is given by:

$$12\pi\eta Ra^2 I(\tfrac{1}{5}k_1-\tfrac{1}{5}k_2+\tfrac{3}{10}l_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

As the dissipation in unit time per unit volume in the undisturbed motion of the liquid has the value [31]):

$$\eta I \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

it is found again that the factor A_{II} must have the value:

$$A_{II}=\frac{9}{8}\frac{a^2}{R^2}\left(\frac{1}{5}k_1-\frac{1}{5}k_2+\frac{3}{10}l_1\right)+2{,}5.$$

(To be continued.)

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[31]) See H. LAMB, Hydrodynamics (Cambridge 1932), p. 580.

Physics. — *The Decrease in the intensity of the cosmic rays in different directions, and the decay of the mesons.* By J. CLAY.

(Communicated at the meeting of March 30, 1940.)

The phenomenon, that the decrease of the intensity of the cosmic radiation in different directions is proportional to the square of the cosine of the angle [1]), which the rays make with the vertical, has recently been explained by the decay of the mesons with a lifetime of about 2.10^{-6} seconds [2]).

This might also be held to explain why in the case of absorption in water, the decrease in the upper layers is smaller than in the deeper layers, where it corresponds to:

$$I = I_0 h^{-1,93}\ ^{3}).$$

FERMI [4]) made the supposition that the abnormal great decrease in the air may lie in the small density of the material in the gaseous state. Assuming a dielectric constant 2 for condensed material, he calculated how it might be explained, that the absorption in water would then be less than what answers to the relative proportion of the mass of water to air, so that the absorption in air for equal quantities would be 1.3 times that of water. In this way the relative decrease in different directions might be explained without presupposing the decay of the mesons.

It is now the question how great this relative coefficient of decrease air/water actually is, and if this is the same for different directions.

According to FERMI's supposition this may be expected, if we take the average density of the air to be the same for all directions.

FERMI, for purposes of comparison, took the relative decrease, from the measurements of EHMERT [5]) as found on the Zugspitze in different directions, as compared with that in water.

It now appeared to us desirable to determine the proportion with other data, for instance, with the aid of the direction mensurements carried out by various authors, also at Amsterdam [1]) and to combine this with measurements we took in different layers of water in Norway [2]), whereby a counter system was chosen so that in both directions the bundle had not a larger opening than 30°.

We have now examined in the first place how the said proportion is found for different directions, assuming the $\cos^2\varphi$ law to be correct, as would appear from the findings of different authors, and when, in the case of the intensities obtained, we see through what depth of water the

rays have to pass in a vertical direction, to decrease to the same intensity. This may be the simplest form of comparison. (Fig. 1.)

In Table 1, column 3 shows the relative intensity for the direction φ at sea-level; in column 4 the thickness a of the layer of air is given in water-

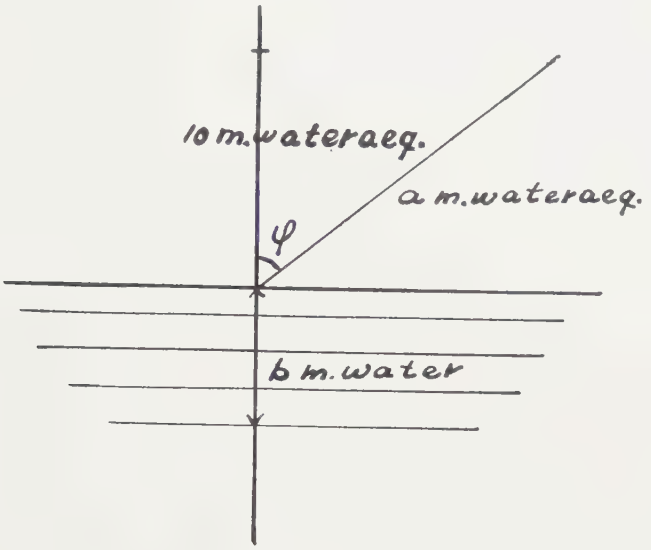

Fig. 1.

equiv.; in column 5 the thickness b is given of the layer of water, through which, according to our results in Norway, a bundle of rays has to pass

TABLE I.

1	2	3	4	5	6
φ	sec α	$\cos^2 \alpha$	$\frac{a}{m}$ wateraeq.	$\frac{b}{m}$ water, where $I/I_0 = \cos^2 \alpha$	$\frac{b}{a-10}$
30°	1.155	0.750	11.55	3	2
40°	1.306	0.586	13.1	6.5	2.1
50°	1.556	0.412	15.5	10.0	1.8
60°	2.00	0.250	20.0	18	1.80
70°	2.93	0.117	29.3	34	1.76
75°	3.77	0.066	37.7	50	1.81
80°	5.78	0.029	57.8	86	1.80

verticaly in order to decrease to the same intensity, and in column 6 the relative value $\frac{b}{a-10}$ is given of this layer in relation to the layer of air, which the rays have passed more than in the vertical direction.

We see that for the thicker layers the proportion of 1.8 is obtained, while it appears that for smaller angles of inclination the value is found to be greater.

With the same apparatus which was used for the water measurements, we again took measurements in different directions in air, and then found

TABLE II.

1	2	3	4	5
φ	a m wateraeq.	I/I_0	b m water $I_w = I$	$\frac{b}{a-10}$
0°	10	1.00		
45°	14.1	0.51	8.0	2.0
60°	20.0	0.27	17	1.7
73°	34.3	0.087	43	1.8

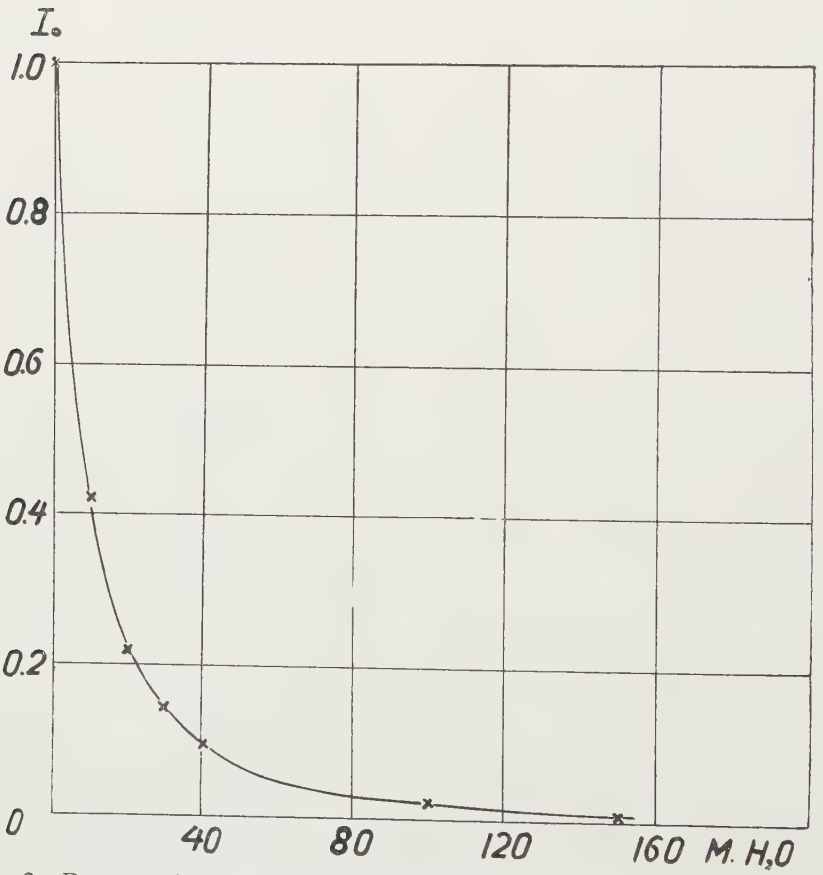

Fig. 2. Decrease of the *vertical* cosmic ratiation in water (J. CLAY, A. v. GEMERT and P. H. CLAY).

the values in Table 2, which agree very well for different directions and are also in accord with the first Table.

Nevertheless it must be borne in mind that for smaller angles a greater value might be found. To compare smaller layers with our apparatus we shall have to take careful measurements at a depth between 0 and 10 m, which up till now has not been practicable, but which will be done as soon as possible.

The value of 1.8, which we obtained, signifies that the value of the dielectric constant would have to be taken appreciably higher than 2, because the formula for the term of the difference contains the logarithmic value, so that it will probably be 10 times greater, as would not be unlikely for water.

Before coming to a definitive conclusion, however, it will be necessary to inquire into what consequences for the decay will have to be inferred with regard to the different directions.

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Mathematics. — *Zur projektiven Differentialgeometrie der Regelflächen im R_4.* (Erste Mitteilung). Von R. WEITZENBÖCK.

(Communicated at the meeting of March 30, 1940.)

Einleitung. Ich stelle in dieser ersten Mitteilung die wichtigsten Formeln zusammen, die für die Untersuchung der projektiv-invarianten Eigenschaften der Umgebung eines allgemeinen Punktes auf einer analytischen Regelfläche F des R_4 die Basis bilden. Die Fläche F erscheint hier gegeben als ∞^1-Schaar von Geraden, ihre Erzeugenden. Dual hierzu ist im R_4 eine einfach-unendliche Schaar von Ebenen; wir sprechen kurz von einer „∞^1-Ebenenschaar F''". Es wird sich später zeigen, dass — abgesehen von dieser Dualität — die Theorie der F' auch in differentialgeometrischer Weise mit der einer Regelfläche F zusammenhängt.

Schliesslich gebe ich hier eine erste projetive Klassifikation der Regelflächen F durch Differentialkomitanten. die die Begriffe: nicht-abwickelbar, abwickelbar, F in einem R_3, Kegelfläche u.s.f. analytisch festlegen.

§ 1.

Die zehn homogenen Koordinaten a_{ik} einer Geraden denken wir uns als analytische Funktionen eines komplexen Parameters t gegeben, sodass — wenigstens in dem Gebiete in dem sich die folgenden Untersuchungen abspielen — die Reihenentwicklung gilt

$$a_{ik}(t) = a_{ik}(0) + \frac{t}{1!}\left(\frac{da_{ik}}{dt}\right)_0 + \frac{t^2}{2!}\left(\frac{d^2 a_{ik}}{dt^2}\right)_0 + \ldots .$$

Die Ableitungen der a_{ik} bezeichnen wir durch

$$a_{ik}, \quad (a_1)_{ik}, \quad (a_2)_{ik}, \ldots .$$

oder auch kürzer durch

$$0_{ik}, \quad 1_{ik}, \quad 2_{ik}, \ldots ,$$

sodass wir auch haben

$$a_{ik}(t) = 0_{ik} + \frac{t}{1!}1_{ik} + \frac{t^2}{2!}2_{ik} + \ldots .$$

Diese Abkürzungen übernehmen wir auch für die Komplexsymbole,

falls solche zur Darstellung der $(a_m)_{ik}$ zur Verwendung kommen. So haben wir beispielsweise für die Kovariante

$$4K = 4\Sigma \pm x_1\, a_{23}\, (a_2)_{45} = 4\Sigma \pm x_1\, 0_{23}\, 2_{45} = \Sigma\, x_i\, (M'_{02})_i \;:$$
$$K = (x\,0^2\,2^2) = (x\,a^2\,a_2{}^2).$$

Da die a_{ik} Linienkoordinaten sind, gilt für alle Parameterwerte t

$$4\Sigma\, x_1\, a_{23}\, a_{45} = (x\,a^2\,0^2) = 0 \quad \{\text{für alle } x_i\}.$$

Wir werden statt $(x\,a^2\,0^2)$ auch $(x\,0^2\,\dot{0}^2)$ schreiben, also $\dot{0}$ äquivalent mit 0 wählen. Die fünf quadratischen Gleichungen für die Linienkoordinaten, a_{ik} die aus $(x\,0^2\,\dot{0}^2) \equiv 0$ folgen, werden wir im Weiteren festhalten durch

$$M'_{00} = 0.$$

Die zehn Grössen $(a_1)_{ik} = 1_{ik}$ sind im Allgemeinen nicht wieder Linienkoordinaten, sondern die Koordinaten eines linearen Ebenenkomplexes, dessen Brennraum durch

$$(x\,a_1{}^2\,1^2) = (x\,1^2\,\dot{1}^2) = 4\Sigma \pm x_1\, (a_1)_{23}\, (a_1)_{45} = 0$$

gegeben wird. Analoges gilt für die

$$(a_m)_{ik}.$$

Aus $M'_{00} = 0$ erhalten wir durch Differentiation nach dem Parameter t eine Reihe von Gleichungen, die wir im Folgenden stets benutzen werden:

$$\left.\begin{array}{rrrrrrrrr}
M'_{00} & = & 0 & & & & & & \\
M'_{11} & + & M'_{02} & = & 0 & & & & \\
3\,M'_{12} & + & M'_{03} & = & 0 & & & & \\
3\,M'_{22} & + & 4\,M'_{13} & + & M'_{04} & = & 0 & & \\
10\,M'_{23} & + & 5\,M'_{14} & + & M'_{05} & = & 0 & & \\
10\,M'_{33} & + & 15\,M'_{24} & + & 6\,M'_{15} & + & M'_{06} & = & 0 \\
35\,M'_{34} & + & 21\,M'_{25} & + & 7\,M'_{16} & + & M'_{07} & = & 0 \\
35\,M'_{44} & + & 56\,M'_{35} & + & 28\,M'_{26} & + & 8\,M'_{17} & + & M'_{08} = 0 \\
126\,M'_{45} & + & 84\,M'_{36} & + & 36\,M'_{27} & + & 9\,M'_{18} & + & M'_{09} = 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}\right\} \quad \ldots \quad (1)$$

Für manche Fragen ist es vorteilhaft die $a_{ik}(t)$ so zu schreiben, dass sie als Linienkoordinaten erkennbar sind. Zu diesem Zwecke denken

wir uns die Erzeugende a_{ik} durch zwei lineare $R_3\, v'$ und w' in den Punkten y und z geschnitten, setzen also

$$y_i = a_i\,(av') \;,\; z_k = a_k\,(aw').$$

Es wird dann

$$(yz)_{ik} = y_i\, z_k - y_k\, z_i = a_{ik} \cdot \Sigma\, a_{rs}\,(v'\, w')_{rs}\,;$$

wählen wir also die linearen $R_3\, v'$ und w' so, dass ihre Schnittebene $(v'\, w')$ die Erzeugende a_{ik} micht schneidet, dann ist

$$\Sigma\, a_{ik}\,(v'\, w')_{ik} \neq 0$$

und wirr können dies $= 1$ setzen, sodass

$$a_{ik} = (yz)_{ik}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

wird, wobei y und z von Parameter t abhängen.

Einen besonderen Platz unter den Regelflächen nehmen die *Torsen* und die *Kegel* ein. Bei einer Torse sind die Erzeugenden die Tangenten einer nicht-ebenen Raumkurve. Ist diese durch $y_i = y_i\,(t)$ gegeben, so haben wir

$$\left.\begin{aligned} 0_{ik} = a_{ik}\,(t) \;&= (yy')_{ik} = (01)_{ik} \\ 1_{ik} = (yy'')_{ik} \;&= (02)_{ik} \\ 2_{ik} = (12)_{ik} \;&+ (03)_{ik} \\ 3_{ik} = 2\,(13)_{ik} &+ (04)_{ik} \\ 4_{ik} = 2\,(23)_{ik} + 3\,(14)_{ik} &+ (05)_{ik} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Und hieraus finden wir für die M'_{rs}:

$$M'_{00} = M'_{01} = M'_{02} = M'_{11} = M'_{12} = M'_{03} = 0,$$
$$M'_{04} = 2\,(yy'\, y''\, y''') = 2\,(0123) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

und dieses letztere ist im Allgemeinen nicht Null, da $(x\,0123) = 0$ den Schmieg-R_3 im Punkte y darstellt.

Bei einer Kegelfläche können wir, wenn die Spitze der feste Punkt s ist, setzen

$$0_{ik} = a_{ik}\,(t) = (y\,s)_{ik}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

woraus für jedes $m = 1, 2, 3, \ldots$ folgt

$$m_{ik} = (a_m)_{ik} = (y^{(m)}\, s)_{ik}\,.$$

Also verschwinden hier alle M'_{ik}.

§ 2.

Als projektive Differentialinvariante J einer durch die ∞^1 Geraden $a_{ik}\,(t)$ im R_4 dargestellten Regelfläche F_2 bezeichnen wir allgemein einen

von den a_{ik}, $(a_1)_{ik}$, ... und eventuellen, vom Parameter t unabhängigen Koordinatenreihen x_i, π_{ik}, π_{ijk}, u_i' abhängigen Ausdruck, der *dreierlei* Invarianz besitzt.

Erstens bei den linear-homogenen Transformationen

$$\varrho\, x_i = e_i^k \bar{x}_k, \quad \triangle = |e_i^k| \neq 0$$

der allgemeinen projektiven Gruppe des R_4:

$$\bar{J} = \triangle^w . J \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Wir werden diese Invarianz weiterhin dadurch als evident voraussetzen, da wir nur mit quinären Invarianten rechnen werden.

Zweitens muss J invariant sein bei Parametertransformationen

$$t = \varphi(\tilde{t}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Man beweist leicht, dass sich bei rationalen Differentialinvarianten diese Invarianz durch die Gleichungen

$$\tilde{J} = (\varphi')^\varrho . J \qquad \varphi' = \frac{d\varphi}{d\tilde{t}} = \frac{dt}{d\tilde{t}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

ausdrückt. Der rationale Exponent ϱ heisst das φ'-Gewicht von J.

Drittens haben wir Invarianz bei den Transformationen

$$\widehat{a_{ik}} = \lambda(t) . a_{ik}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

wo also alle a_{ik} mit demselben Faktor λ multipliziert erscheinen. Hier gilt dann

$$\widehat{J} = \lambda^\sigma . J \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

und σ heisst das λ-Gewicht der Invariante J.

Bei den Transformationen (7) hat man:

$$\left.\begin{aligned}
&\widetilde{0}_{ik} = 0_{ik} \\
&\widetilde{1}_{ik} = 1_{ik} . \varphi' \\
&\widetilde{2}_{ik} = 2_{ik} . \varphi'^2 + 1_{ik} . \varphi'' \\
&\widetilde{3}_{ik} = 3_{ik} . \varphi'^3 + 2_{ik} . \ 3\varphi' \varphi'' + 1_{ik} . \varphi''' \\
&\widetilde{4}_{ik} = 4_{ik} . \varphi'^4 + 3_{ik} . \ 6\varphi'^2 \varphi'' + 2_{ik} . (3\varphi''^2 + 4\varphi' \varphi''') + 1_{ik} . \varphi^{IV} \\
&\widetilde{5}_{ik} = 5_{ik} . \varphi'^5 + 4_{ik} . 10\varphi'^3 \varphi'' + 3_{ik} . (15\varphi' \varphi''^2 + 10\varphi'^2 \varphi''') + \\
&\qquad\qquad + 2_{ik} . (10\varphi'' \varphi''' + 5\varphi' \varphi^{IV}) + 1_{ik} . \varphi^V
\end{aligned}\right\} . \quad (11)$$

. . . . . .

Hieraus folgen für die M'_{rs}:

$$\left.\begin{aligned}
&\widetilde{M}'_{02} = -\widetilde{M}'_{11} = M'_{02} \,.\, \varphi'^2 \\
&\widetilde{M}'_{03} = -3\widetilde{M}'_{12} = M'_{03} \,.\, \varphi'^3 + M'_{02} \,.\, 3\varphi'\,\varphi'' \\
&\widetilde{M}'_{04} = M'_{04} \,.\, \varphi'^4 + M'_{03} \,.\, 6\varphi'^2\,\varphi'' + M'_{02} \,.\, (3\varphi''^2 + 4\varphi'\,\varphi''') \\
&\widetilde{M}'_{13} = M'_{13} \,.\, \varphi'^4 - M'_{03} \,.\, \varphi'^2\,\varphi'' - M'_{02} \,.\, \varphi'\,\varphi''' \\
&\widetilde{M}'_{22} = M'_{22} \,.\, \varphi'^4 - M'_{03} \,.\, \tfrac{2}{3}\,\varphi'^2\,\varphi'' - M'_{02} \,.\, \varphi''^2 \\
&\widetilde{M}'_{05} = M'_{05} \,.\, \varphi'^5 + M'_{04} \,.\, 10\varphi'^3\,\varphi'' + M'_{03} \,.\, (15\varphi'\,\varphi''^2 + \\
&\qquad\qquad + 10\varphi'^2\,\varphi''') + M'_{02} \,.\, (10\varphi''\,\varphi''' + 5\varphi'\,\varphi^{IV}) \\
&\widetilde{M}'_{23} = M'_{23} \,.\, \varphi'^5 - M'_{13} \,.\, 3\varphi'^3\,\varphi'' - M'_{04} \,.\, \varphi'^3\,\varphi'' - M'_{03} \,.\, (\varphi'\,\varphi''^2 + \\
&\qquad\qquad + \tfrac{1}{3}\,\varphi'^2\,\varphi''') - M'_{02} \,.\, \varphi''\,\varphi''' \\
&\widetilde{M}'_{33} = M'_{33} \,.\, \varphi'^6 + M'_{23} \,.\, 6\varphi'^4\,\varphi'' + M'_{13} \,.\, (2\varphi'^3\,\varphi''' - 12\,\varphi'^2\,\varphi''^2) - \\
&\qquad\qquad - M'_{04} \,.\, 3\varphi'^2\,\varphi''^2 - M'_{03} \,.\, 2\varphi'\,\varphi''\,\varphi''' - M'_{02} \,.\, \varphi'''^2 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \quad (12)$$

Die entsprechenden Formeln bei den Transformationen (9) lauten:

$$\left.\begin{aligned}
&\widehat{0}_{ik} = 0_{ik} \,.\, \lambda \\
&\widehat{1}_{ik} = 1_{ik} \,.\, \lambda + 0_{ik} \,.\, \lambda' \\
&\widehat{2}_{ik} = 2_{ik} \,.\, \lambda + 1_{ik} \,.\, 2\lambda' + 0_{ik} \,.\, \lambda'' \\
&\widehat{3}_{ik} = 3_{ik} \,.\, \lambda + 2_{ik} \,.\, 3\lambda' + 1_{ik} \,.\, 3\lambda'' + 0_{ik} \,.\, \lambda''' \\
&\widehat{4}_{ik} = 4_{ik} \,.\, \lambda + 3_{ik} \,.\, 4\lambda' + 2_{ik} \,.\, 6\lambda'' + 1_{ik} \,.\, 4\lambda''' + 0_{ik} \,.\, \lambda^{IV} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \quad (13)$$

Dies gibt für die $\widehat{M}'_{rs}$:

$$\left.\begin{aligned}
&\widehat{M}'_{02} = M'_{02} \,.\, \lambda^2 \\
&\widehat{M}'_{03} = M'_{03} \,.\, \lambda^2 + M'_{02} \,.\, 3\lambda\lambda' \\
&\widehat{M}'_{04} = M'_{04} \,.\, \lambda^2 + M'_{03} \,.\, 4\lambda\lambda' + M'_{02} \,.\, 6\lambda\lambda'' \\
&\widehat{M}'_{13} = M'_{13} \,.\, \lambda^2 + M'_{02} \,.\, (3\lambda'^2 - 3\lambda\lambda'') \\
&\widehat{M}'_{22} = M'_{22} \,.\, \lambda^2 - M'_{03} \,.\, \tfrac{4}{3}\,\lambda\lambda' + M'_{02} \,.\, (2\lambda\lambda'' - 4\lambda'^2) \\
&\widehat{M}'_{05} = M'_{05} \,.\, \lambda^2 + M'_{04} \,.\, 5\lambda\lambda' + M'_{03} \,.\, 10\lambda\lambda'' + M'_{02} \,.\, 5\lambda\lambda''' \\
&\widehat{M}'_{23} = M'_{23} \,.\, \lambda^2 - M'_{13} \,.\, 2\lambda\lambda' - M'_{04} \,.\, \lambda\lambda' - M'_{03} \,.\, 2\lambda'^2 + M'_{02} (\lambda\lambda''' - 3\lambda'\,\lambda'') \\
&\widehat{M}'_{33} = M'_{33} \,.\, \lambda^2 + M'_{23} \,.\, 6\lambda\lambda' + M'_{13} \,.\, (6\lambda\lambda'' - 12\lambda'^2) - M'_{04} \,.\, 3\lambda'^2 + \\
&\qquad\qquad + M'_{03} \,.\, (2\lambda\lambda''' - 6\lambda'\,\lambda'') + M'_{02} \,.\, (6\lambda'\,\lambda''' - 9\lambda''^2) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \quad (14)$$

Beispiele.

Die quinäre Linearform

$$J = (x M'_{02}) = (x\, 0^2\, 2^2) = 4 \Sigma \pm x_1\, a_{23}\, (a_2)_{45} \quad . \quad . \quad . \quad . \quad (15)$$

ist eine Differentialkovariante von F von φ'-Gewichte und λ-Gewichte zwei:

$$\widetilde{J} = \varphi'^2 \,.\, J \quad , \quad \widehat{J} = \lambda^2 \,.\, J.$$

2. Die Linearform in R_3-Koordinaten u'_i

$$(H u') = H = (0\, u')\, (0\, M'_{22}) = (0\, u')\, 0_{22} = \Sigma\, a_{ik}\, (u'\, M'_{22})_{ik} \,. \quad . \quad (16)$$

ist eine Differentialkontravariante vom φ'-Gewichte 4 und λ-Gewicht 3:

$$\widetilde{H} = \varphi'^4 \,.\, H \quad , \quad \widehat{H} = \lambda^3 \,.\, H.$$

3. Der Ausdruck

$$Q = 0_{13,\,23} = (a\, M'_{13})\, (a\, M'_{23}) = \Sigma\, a_{ik}\, (M'_{13}\, M'_{23})_{ik} \quad . \quad . \quad . \quad (17)$$

ist eine Differentialinvariante vom φ'-Gewicht 9 und λ-Gewicht 5:

$$\widetilde{Q} = \varphi'^9 \,.\, Q \quad , \quad \widehat{Q} = \lambda^5 \,.\, Q,$$

was man auf Grund der Formeln (11) bis (14) leicht nachrechnet.

Man beweist leicht den Satz[1]: *Sind J_1, J_2, J_3 drei Differentialinvarianten mit den Gewichten ϱ_i, σ_i, so ist*

$$\{J_1,\, J_2,\, J_3\} = \begin{vmatrix} J'_1 & J'_2 & J'_3 \\ \varrho_1 J_1 & \varrho_2 J_2 & \varrho_3 J_3 \\ \sigma_1 J_1 & \sigma_2 J_2 & \sigma_3 J_3 \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

eine Differentialinvariante vom φ'-Gewicht $\varrho_1 + \varrho_2 + \varrho_3 + 1$ und vom λ-Gewicht $\sigma_1 + \sigma_2 + \sigma_3$.

Bei den eben genannten drei Komitanten z.B. ergibt sich nach diesem Satze der Konnex

$$K = \begin{vmatrix} (x M'_{12}) + (x M'_{03}) & (H_1\, u') & Q' \\ 2\, (x M'_{02}) & 4\, (H u') & 9\, Q \\ 2\, (x M'_{02}) & 3\, (H u') & 5\, Q \end{vmatrix},$$

oder, entwickelt:

$$K = 3\, (x M'_{02}) \,.\, [4\, Q\, (H_1\, u') - Q'\, (H u')] - 7\, (x M'_{03})\, Q\, (H u'), \quad . \quad (19)$$

wobei

$$(H_1\, u') = \frac{d\,(H u')}{dt} = (1\, u')\, 1_{22} + 2 \,.\, (0\, u')\, 0_{23}$$

gesetzt ist.

[1]) Für die Vm im Rn vgl. diese Proceedings 27 734—741, (1924).

§ 3.

Wir sahen bei den Gleichungen (3) und (5), dass bei einer Torse und bei einer Kegelfläche die Kovariante M'_{02} verschwindet. $(x M'_{02}) = 0$ stellt den dreidimensionalen Tangentialraum an F längs der Erzeugenden a_{ik} dar. Jetzt wollen wir hiervon die Umkehrung beweisen, d.h. genauer, zeigen, dass durch

$$(x M'_{02}) \equiv 0 \qquad \{\text{für alle } x_i\}$$

die *abwickelbaren Flächen* karakterisiert sind. Hierbei nennen wir eine geradlinige Fläche F abwickelbar, wenn jede Erzeugende von ihrer benachbarten geschnitten wird. Es fallen also unter diesen Begriff z.B. die Gesamtheit der Tangenten einer ebenen Kurve oder die Geraden eines Geradenbüschels. Den Fall $a_{ik} = \text{constant}$ schliessen wir natürlich aus, da dann keine Fläche F vorliegt.

Wenn wir a_{ik} in der Gestalt (2) annehmen, wobei die Punkte y und z zwei Kurven C_y bzw. C_z auf F durchlaufen, so gilt, wenn die Akzente das Differentieren nach t bedeuten:

$$a_{ik} = (yz)_{ik} \ , \ (a_1)_{ik} = (y'z)_{ik} + (yz')_{ik}$$

und die Voraussetzung

$$(x M'_{02}) = -(x M'_{11}) \equiv 0$$

geht dann über in

$$(x M'_{11}) = (y' z y z' x) \equiv 0 \quad \{\text{in allen } x_i\} \quad . \ . \ . \ . \ (20)$$

Andererseits haben wir für die Koordinaten der zu a_{ik} benachbarten Erzeugenden

$$(y + y'\,dt,\ z + z'\,dt)_{ik} = (yz)_{ik} + [(yz')_{ik} + (y'z)_{ik}]\,.\,dt + (y'z')\,dt^2$$

und dass diese $a_{ik} = (yz)_{ik}$ schneidet, führt auf (20).

Bei den *nicht-abwickelbaren Flächen* F unterscheiden wir dann zwei Klassen IA und IB. Die zu IA gehörigen Flächen F liegen nicht in einem R_3, die von IB liegen in einem linearen dreidimensionalen Raume. Wir beweisen jetzt, dass IA karakterisiert wird durch

$$H = (Hu') = (au')(aa_2^2 b_2^2) = (au')\,(aM'_{22}) = (0u')\,0_{22} \not\equiv 0; \ . \ . \ (21)$$

dass dagegen bei IB diese Kontravariante (Hu') für alle u'_i verschwindet, dass also $H \not\equiv 0$ notwendig und hinreichend dafür ist, dass F in einem linearen R_3 liegt.

Beweis. Sei zunächst F in einem R_3; dann ist wegen $M'_{02} \not\equiv 0$ nach (20)

$$(yzy'z'x) \not\equiv 0$$

und der R_3 ist durch $(yzy'z')_i$ gegeben. Die Punkte y'' und z'' gehören

dann ebenfalls diesem R_3 an, da sie in den Schmiegebenen von C_y und C_z liegen. Es gilt also

$$\left.\begin{aligned} y'' &= a\ \ y + b\,.\,z + c\,.\,y' + d\,.\,z' \\ z'' &= A\,.\,y + B\,.\,z + C\,.\,y' + D\,.\,z' \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Berechnet man jetzt vermöge

$$\left.\begin{aligned} a_{ik} &= (yz)_{ik} \\ (a_1)_{ik} &= (y'z)_{ik} + (yz')_{ik} \\ (a_2)_{ik} &= (y''z)_{ik} + 2(y'z')_{ik} + (yz'')_{ik} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (23)$$

die Kontravariante (Hu'), so findet man

$$(Hu') = (yzy'z'y'')\,(zu') - (yzy'z'z'')\,(yu'), \quad . \quad . \quad . \quad . \quad (24)$$

also nach (22) $H \equiv 0$.

Gehen wir umgekehrt von $(Hu') \equiv 0$ aus, so zeigt (24), dass $(yzy'z'y'')$ und $(yzy'z'z'')$ Null sein müssen, woraus sich wieder (22) ergibt.

Bei den *abwickelbaren Regelflächen* gilt $M'_{02} \equiv 0$ und die Kontravariante H von (21) ist stets Null wegen

$$(Hu') = (0u')\,0_{22} = -2\,.\,(2u')\,2_{02} = -2\,.\,(a_2 u')\,(a_2\, a^2\, b_2{}^2).$$

Auch hier unterscheiden wir vorerst zwei Klassen IIA und IIB, je nachdem F dem R_4 oder aber einem linearen R_3 angehört. Die Entscheidung hierüber gibt die Komitante

$$U^* = (a^2 a_1 a_2 a_3)\,(a_1 u')\,(a_2 v')\,(a_3 w') = (0^2\,123)\,(1u')\,(2v')\,(3w') \quad . \quad (25)$$

Berechnen wir U^* für eine Torse mit $a_{ik} = (yy')_{ik}$, so kommt

$$U^* = (yy'y''y'''y^{IV})\,.\,(u'y)\,(v'y)\,(w'y) \quad . \quad . \quad . \quad . \quad . \quad (26)$$

und bei einem Kegel $a_{ik} = (ys)_{ik}$ wird

$$U^* = (yy'y''y'''s)\,.\,(u's)\,(v's)\,(w's) \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Liegt F in einem R_3, so entsteht in beiden Fallen $U^* \equiv 0$. Umgekehrt erhält man bei (26) aus $U^* \equiv 0$ das Verschwinden von $(yy'y''y'''y^{IV})$, also liegt die Kurve C_y und damit F in einem R_3. Bei (27) dagegen haben wir bei $U^* \equiv 0$:

$$(yy'y''y'''s) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

also

$$s = \alpha\,.\,y + \beta\,.\,y' + \gamma\,.\,y'' + \delta\,.\,y'''.$$

Differenziert man dies nach t, so ergibt sich eine lineare Abhängigkeit für y^{IV} von y, y', y'' und y'''; also liegt C_y in einem R_3, dem wegen (28) auch die Spitze s des Kegels angehört.

Die weitere Entscheidung, ob man mit einer Torse oder mit einem Kegel zu tun hat, gibt die Differentialkomitante (mit veränderlichen Linienkoordinaten π_{ik} und ϱ_{ik})

$$V^* = (01^2\,\pi^2)\,(02^2\,\varrho^2) = (aa_1{}^2\,\pi^2)\,(aa_2{}^2\,\varrho^2) \quad . \quad . \quad . \quad . \quad (29)$$

$V^* \not\equiv 0$ gibt die Torsen. $V^* \equiv 0$ die Kegel. Dass V^* bei Kegeln, also für $a_{ik} = (ys)_{ik}$, $s = \text{const.}$, verschwindet, rechnet man leicht nach. Bei $a_{ik} = (yy')_{ik}$ wird

$$V^* = (yy'\,y''\,\pi^2)\,.\,(yy'\,y''\,\varrho^2).$$

Wäre dies für alle π_{ik} und ϱ_{ik} Null, so würde

$$y'' = \alpha\,.\,y + \beta\,y' \quad , \quad C_y = \text{Gerade} \quad , \quad a_{ik} = \text{const.}$$

folgen, also keine Fläche F vorliegen.

Bei den abwickelbaren Flächen F mit $U^* \equiv 0$, die also in einem R_3 liegen, gibt schliesslich neben V^* noch die Komitante

$$W^* = (012^2\,x)\,(0\,v')\,(1\,w') = (aa_1\,a_2{}^2\,x)\,(av')\,(a_1\,w') \quad . \quad . \quad (30)$$

eine letzte Einteilung: $W^* \not\equiv 0$, F ist nicht eben; $W^* \equiv 0$, F ist eben. Berechnet man W^* für $a_{ik} = (yy')_{ik}$ so entsteht

$$W^* = (yy'\,y''\,y'''\,x)\,.\,(y\,v')\,(y\,w')$$

und $W^* \equiv 0$ ist für eine ebene Kurve C_y notwendig und hinreichend.

Bei einem Kegel mit $a_{ik} = (ys)_{ik}$ haben wir

$$W^* = (yy'\,y''\,sx)\,.\,(sv')\,(sw')$$

und auch hier ist $W^* \equiv 0$ für einen ebenen Kegel, d.h. für ein Geradenbüschel notwendig und hinreichend.

Wir stellen nach Obigem für eine erste Klassifikation der analytischen Regelflächen des R_4 vom projektiv-differentialgeometrischen Standpunkte aus, folgende Tabelle zusammen:

- $M'_{02} \not\equiv 0$, nicht-abwickelbare F
  - $H \not\equiv 0$, allgemeine Fläche
  - $H \equiv 0$, F in einem R_3
- $M'_{02} \equiv 0$, abwickelbare Fläche F
  - $U^* \not\equiv 0$, F im R_4
    - $V^* \not\equiv 0$, Torse des R_4
    - $V^* \equiv 0$, Kegel des R_3
  - $U^* \equiv 0$, F im R_3
    - $V^* \not\equiv 0$, $W^* \not\equiv 0$, Torse im R_3
    - $V^* \not\equiv 0$, $W^* \equiv 0$, F gebildet von den Tangenten einer ebenen Kurve
    - $V^* \equiv 0$, $W^* \not\equiv 0$, Kegel im R_3
    - $V^* \equiv 0$, $W^* \equiv 0$, Geradenbüschel

Mathematics. — *Ueber Differentialkomitanten zweier kontravarianter Grössen* [1]). Von J. A. SCHOUTEN.

(Communicated at the meeting of March 30, 1940.)

Bekanntlich lässt sich aus einem kontravarianten Vektor v^ν und einem beliebigen Affinor $P^{\nu_1 \ldots \nu_p}_{\cdot\cdot\cdot\ \lambda_1 \ldots \lambda_q}$ in folgender Weise eine Differentialkomitante bilden

$$Q^{\nu_1 \ldots \nu_p}_{\cdot\cdot\cdot\ \lambda_1 \ldots \lambda_q} = v^\mu \partial_\mu P^{\nu_1 \ldots \nu_p}_{\cdot\cdot\cdot\ \lambda_1 \ldots \lambda_q} - \sum_i^{1,\ldots,p} P^{\nu_1 \ldots \nu_{i-1}\, \mu \nu_{i+1} \ldots \nu_p}_{\cdot\cdot\cdot\ \lambda_1 \ldots \lambda_q} \partial_\mu v^{\nu_i} + \sum_j^{1,\ldots,q} P^{\nu_1 \ldots \nu_p}_{\cdot\cdot\cdot\ \lambda_1 \ldots \lambda_{j-1}\, \mu \lambda_{j+1} \ldots \lambda_q} \partial_{\lambda_q} v^\mu \quad (1)$$

Dass dieser Ausdruck tatsächlich ein Affinor ist, folgt daraus, dass

$$Q^{\nu_1 \ldots \nu_p}_{\cdot\cdot\cdot\ \lambda_1 \ldots \lambda_q} = v^\mu \nabla_\mu P^{\nu_1 \ldots \nu_p}_{\cdot\cdot\cdot\ \lambda_1 \ldots \lambda_q} - \sum_i^{1,\ldots,p} P^{\nu_1 \ldots \nu_{i-1}\, \mu \nu_{i+1} \ldots \nu_p}_{\cdot\cdot\cdot\ \lambda_1 \ldots \lambda_q} \overleftarrow{\nabla}_\mu v^{\nu_i} + \sum_j^{1,\ldots,q} P^{\nu_1 \ldots \nu_p}_{\cdot\cdot\cdot\ \lambda_1 \ldots \lambda_{j-1}\, \mu \lambda_{j+1} \ldots \lambda_q} \overleftarrow{\nabla}_{\lambda_j} v^\mu \quad (2)$$

ist, wo ∇ das Symbol der kovarianten Ableitung einer linearen Uebertragung mit *beliebigen* Parametern $\Gamma^\varkappa_{\mu\lambda}$ ist und $\overleftarrow{\nabla}$ das Symbol der kovarianten Ableitung mit den Parametern ${}'\Gamma^\varkappa_{\mu\lambda} = \Gamma^\varkappa_{\lambda\mu}$. Der Beweis erfolgt, indem man (2) ausschreibt und feststellt, dass alle Glieder mit $\Gamma^\varkappa_{\mu\lambda}$ sich gegenseitig aufheben. Der Ausdruck

$$\underset{L}{\delta} P^{\nu_1 \ldots \nu_p}_{\cdot\cdot\cdot\ \lambda_1 \ldots \lambda_q} = R^{\nu_1 \ldots \nu_p}_{\cdot\cdot\cdot\ \lambda_1 \ldots \lambda_q}\, dt \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

ist das LIE'sche Differential von $P^{\nu_1 \ldots \nu_p}_{\cdot\cdot\cdot\ \lambda_1 \ldots \lambda_q}$ in bezug auf die infinitesimale Verrückung $v^\nu\, dt$. Dieser Differentialausdruck ist 1931 von W. SLEBODZINSKI [2]) gebildet und hat seitdem vielfach Anwendung

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[1]) Die Differentialinvarianten kovarianter Grössen sind behandelt von R. WEITZENBÖCK in Ueber Differentialinvarianten von kovarianten Tensoren, Proc. Kon. Akad. v. Wetensch., Amsterdam, **29**, 400—403 (1926).

[2]) W. SLEBODZINSKI, Sur les équations canoniques de HAMILTON. Bull. Acad. Royale de Belgique (5) **17**, 864—870 (1931).

gefunden, insbesondere bei Variationsproblemen. Geometrisch bedeutet das LIE'sche Differential die Differenz zwischen dem wirklichen Feldwert und dem über $v^\varkappa\, dt$ „mitgeschleppten" Feldwert [1]).

Es seien nun zwei kontravariante Affinoren $P^{\mu\varkappa_1\cdots\varkappa_a}$ und $Q^{\lambda\varkappa_1\cdots\varkappa_b}$ von der Valenz $a+1$ bzw. $b+1$ gegeben. Dann gilt die Gleichung

$$\left.\begin{aligned} R^{\varkappa_1\cdots\varkappa_{a+b+1}} &\overset{\text{def}}{=} \sum_i^{0,\ldots,a} \mathfrak{P}^i\, P^{\{\varkappa_1\cdots\varkappa_i\,|\,\mu\,|\,\varkappa_{i+1}\cdots\varkappa_a}\, \partial_\mu\, Q^{\varkappa_{a+1}\cdots\varkappa_{a+b+1}\}} \\ &- \sum_j^{0,\ldots,b} \mathfrak{P}^{ab+a+b+j} Q^{\{\varkappa_1\ldots\varkappa_j\,|\,\lambda\,|\,\varkappa_{j+1}\cdots\varkappa_b}\, \partial_\lambda\, P^{\varkappa_{b+1}\cdots\varkappa_{a+b+1}\}} = \\ &= \sum_i^{0,\ldots,a} \mathfrak{P}^i\, P^{\{\varkappa_1\ldots\varkappa_i\,|\,\mu\,|\,\varkappa_{i+1}\cdots\varkappa_a}\, \nabla_\mu\, Q^{\varkappa_{a+1}\cdots\varkappa_{a+b+1}\}} \\ &- \sum_j^{0,\ldots,b} \mathfrak{P}^{ab+a+b+j}\, Q^{\{\varkappa_1\ldots\varkappa_j\,|\,\lambda\,|\,\varkappa_{j+1}\cdots\varkappa_b}\, \overleftarrow{\nabla}_\lambda\, P^{\varkappa_{b+1}\cdots\varkappa_{a+b+1}\}} \end{aligned}\right\} . \quad (4)$$

wo $\{\,\}$ bedeutet, dass die Summe aller *geraden* Permutationen der eingeklammerten Indizes genommen und durch deren Anzahl dividiert werden soll (sodass also $\{\,\} = (\,) + [\,]$ ist) und $\mathfrak{P}^i$ ein gerader oder ungerader Permutationsoperator dieser Indizes ist für i gerade bzw. ungerade. Es ist also z.B. $\mathfrak{P}\{\varkappa\lambda\nu\} = \{\lambda\varkappa\nu\}$ und $\mathfrak{P}^2\{\varkappa\lambda\nu\} = \{\varkappa\lambda\nu\}$. ∇ und $\overleftarrow{\nabla}$ haben dieselbe Bedeutung wie oben.

Beweis.

Schreibt man die rechte Seite der Gleichung aus, so fallen die Terme, die $\Gamma^\varkappa_{\mu\lambda}$ nicht enthalten, gegen die linke Seite fort und es bleibt

$$\left.\begin{aligned} &\sum_i^{0,\ldots,a} \sum_j^{0,\ldots,b} \mathfrak{P}^i\, P^{\{\varkappa_1\ldots\varkappa_i|\mu|\varkappa_{i+1}\cdots\varkappa_a}\, \Gamma^{\varkappa_{a+j+1}}_{\mu\lambda}\, Q^{\varkappa_{a+1}\cdots\varkappa_{a+j}|\lambda|\varkappa_{a+j+2}\cdots\varkappa_{a+b+1}\}} - \\ &- \sum_j^{0,\ldots,b} \sum_i^{0,\ldots,a} \mathfrak{P}^{ab+a+b+j}\, Q^{\{\varkappa_1\ldots\varkappa_j|\lambda|\varkappa_{j+1}\cdots\varkappa_b}\, \Gamma^{\varkappa_{b+i+1}}_{\mu\lambda}\, P^{\varkappa_{b+1}\cdots\varkappa_{b+i}|\mu|\varkappa_{b+i+2}\cdots\varkappa_{a+b+1}\}} = \\ &= \sum_i^{0,\ldots,a} \sum_j^{0,\ldots,b} \mathfrak{P}^{i+j+b}\, P^{\{\varkappa_1\ldots\varkappa_i|\mu|\varkappa_{i+1}\cdots\varkappa_a}\, Q^{\varkappa_{a+1}\cdots\varkappa_{a+j}|\lambda|\varkappa_{a+j+1}\cdots\varkappa_{a+b}}\, \Gamma^{\varkappa_{a+b+1}\}}_{\mu\lambda} - \\ &- \sum_j^{0,\ldots,b} \sum_i^{0,\ldots,a} \mathfrak{P}^{i+j+b}\, P^{\{\varkappa_1\ldots\varkappa_i|\mu\,\varkappa_{i+1}\cdots\varkappa_a}\, Q^{\varkappa_{a+1}\cdots\varkappa_{a+j}|\lambda\,\varkappa_{a+j+1}\cdots\varkappa_{a+b}}\, \Gamma^{\varkappa_{a+b+1}\}}_{\mu\lambda} = 0 \end{aligned}\right\} . \quad (5)$$

womit der Beweis geliefert ist.

Da R also ein Affinor ist, ist eine Differentialkomitante von der Valenz

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[1]) Vergl. Einführung in die neueren Methoden der Differentialgeometrie von J. A. SCHOUTEN und D. J. STRUIK, I 142, II 161.

$a+b+1$ entstanden, die sich bei Zerlegung von $\{\}$ in $()$ und $[]$ auch folgendermassen schreiben lässt:

$$\left.\begin{aligned} R^{z_1\cdots z_{a+b+1}} = & \sum_{i}^{0,\ldots,a} P^{(z_1\cdots z_i\,|\mu|\,z_{i+1}\cdots z_a}\,\partial_\mu\,Q^{z_{a+1}\cdots z_{a+b+1})} \\ & + \sum_{i}^{0,\ldots,a} (-1)^i\,P^{[z_1\cdots z_i\,\mu|\,z_{i+1}\cdots z_a}\partial_\mu\,Q^{z_{a+1}\cdots z_{a+b+1}]} \\ & - \sum_{j}^{0,\ldots,b} Q^{(z_1\cdots z_j\,|\lambda|\,z_{j+1}\cdots z_b}\,\partial_\lambda\,P^{z_{b+1}\cdots z_{a+b+1})} \\ & - \sum_{j}^{0,\ldots,b} (-1)^{ab+a+b+j}\,Q^{[z_1\cdots z_j\,\lambda|\,z_{j+1}\cdots z_b}\,\partial_\lambda\,P^{z_{b+1}\cdots z_{a+b+1}]} \end{aligned}\right\}. \quad (6)$$

in welchem Ausdruck natürlich ebenfalls ∂_μ durch ∇_μ und (gleichzeitig!) ∂_λ durch $\overleftarrow{\nabla}_\lambda$ ersetzt werden darf.

Die gefundene Differentialkomitante hat einige merkwürdige Eigenschaften.

1. Ist $a=0$, so geht R nur für $b=0$ und für $b=1$ über in die Komitante, die sich nach der Vorschrift (1) bilden lässt. Für $a=0$ und $b>1$ entsteht die Summe des alternierenden und des symmetrischen Teiles letzterer Komitante. Die neue Komitante bringt also erst für $a>0$, $b>0$ Neues.

2. Ist $P(Q)$ symmetrisch bzw. alternierend, so geht in R für $a>0$, $b>0$ nur der symmetrische bzw. alternierende Teil von $Q(P)$ ein. Ist dieser Teil also Null, so ist auch R Null. Insbesondere ist also die Komitante einer symmetrischen und einer alternierenden Grösse Null.

3. Zwei symmetrische (alternierende) Grössen bilden für $a>0, b>0$ eine symmetrische (alternierende) Komitante. Statt (6) kommt für symmetrische Grössen einfacher

$$\left.\begin{aligned} R^{z_1\cdots z_{a+b+1}} = (a+1)\,P^{\mu(z_1\cdots z_a}\,\partial_\mu\,Q^{z_{a+1}\cdots z_{a+b+1})} - \\ -(b+1)\,Q^{\mu(z_1\cdots z_b}\,\partial_\mu\,P^{z_{b+1}\cdots z_{a+b+1})} \end{aligned}\right\}. \quad (7)$$

und für alternierende Grössen

$$\left.\begin{aligned} R^{z_1\cdots z_{a+b+1}} = (a+1)\,P^{\mu[z_1\cdots z_a}\,\partial_\mu\,Q^{z_{a+1}\cdots z_{a+b+1}]} - \\ -(-1)^{ab+a+b}\,(b+1)\,Q^{\mu[z_1\cdots z_b}\,\partial_\mu\,P^{z_{b+1}\cdots z_{a+b+1}]} \end{aligned}\right\}. \quad (8)$$

Beide Formeln gelten auch wenn $a=0$ $(b=0)$ und $Q(P)$ für $b>0$ $(a>0)$ symmetrisch bzw. alternierend ist. R ist dann also auch symmetrisch bzw. alternierend.

4. Genügt ein Vektor w_λ einem System von Gleichungen

$$X_a^{\mu\varkappa_1\ldots\varkappa_{s_a}}\, w_\mu\, w_{\varkappa_1}\ldots . w_{\varkappa_{s_a}} = 0\ ;\ a = 1, \ldots, p\ .\ .\ .\ .\ (9)$$

wo die $X_a^{\mu\varkappa_1\ldots\varkappa_{s_a}}$ symmetrisch sind, und schreibt man $X_{ab}^{\mu\varkappa_1\ldots\varkappa_{s_a+s_b}}$ für die Komitante von X_a und X_b, so beweist man leicht, dass

$$\left.\begin{aligned} &X_{ab}^{\mu\varkappa_1\ldots\varkappa_{s_a+s_b}}\, w_\mu\, w_{\varkappa_1}\ldots . w_{\varkappa_{s_a+s_b}} = \\ &= 2\,(s_a+1)\,(s_b+1)\, X_a^{\lambda\varkappa_1\ldots\varkappa_{s_a}}\, X_b^{\mu\nu_1\ldots\nu_{s_b}}\, w_{\varkappa_1}\ldots . w_{\varkappa_{s_a}}\, w_{\nu_1}\ldots . w_{\nu_{s_b}}\, \partial_{[\mu} w_{\lambda]} \end{aligned}\right\}\ .\ (10)$$

ist, in welcher Gleichung man rechts wegen der Symmetrie der X statt über $\mu\lambda$ auch über $a\,b$ alternieren darf. Sind die Valenzen der X_a alle eins und ist gefordert dass $\partial_{[\mu} w_{\lambda]} = 0$ sei, also $w_\lambda = \partial_\lambda p$, so ist (9) ein System linearer partieller Differentialgleichungen erster Ordnung in p und man benutzt dann die Beziehung (10) bekanntlich dazu um das System (9) zu verlängern und zu einem vollständigen System zu gelangen. Ueber einen ähnlichen Verlängerungsprozess der Gleichungen (9) mittels (10) für allgemeine Werte der Valenzen, wenn allgemeinere Forderungen für $\partial_{[\mu} w_{\lambda]}$ aufgestellt werden, soll in einer folgenden Mitteilung berichtet werden.

Mathematics. — *Beiträge zur Theorie der Systeme* PFAFFscher *Gleichungen. III. Beweis des Haupttheorems für den Fall dass der Rang den höchsten Wert hat.* Von J. A. SCHOUTEN und W. VAN DER KULK.

(Communicated at the meeting of March 30, 1940.)

1. *Formulierung des Haupttheorems.*

a. Es sei

$$C_\lambda^x \, dx^\lambda = 0 \; ; \; x = p+1, \ldots, n \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

ein System von $q = n-p$ linear unabhängigen PFAFFschen Gleichungen in n Variabeln und

$$B_b^\varkappa \, \partial_\varkappa f = 0 \; ; \; b = 1, \ldots, p \; ; \; B_b^\mu \, C_\mu^x = 0. \quad . \quad . \quad . \quad . \quad (2)$$

das adjungierte System von p linear unabhängigen partiellen Differentialgleichungen und es seien die C_λ^x und $B_b^\varkappa$ analytisch in der Umgebung des beliebig gewählten Punktes $x^\varkappa = \underset{0}{x}{}^\varkappa$.

Ferner sei

$$C_{[a_1 b_1}^{\quad\cdot\quad\cdot\quad x_1} \ldots . \, C_{a_\sigma b_\sigma]}^{\quad\cdot\quad\cdot\quad x_\sigma} \begin{cases} = 0 \text{ für } \sigma > \varrho \\ \neq 0 \text{ für } \sigma = \varrho \end{cases} ; \; C_{ab}^{\cdot\cdot x} = 2 \, C_\lambda^x \, B_{[a}^\mu \, \partial_{|\mu|} \, B_{b]}^\lambda . \quad . \quad (3)$$

d.h. es sei der Halbrang (nach ENGEL) von (1) gleich ϱ. $(2\varrho \leqq p)$. Sodann besitzt das System von $p-\varrho$ multilinearen Differentialgleichungen mit $\varrho+1$ Unbekannten

$$\left\| \begin{matrix} X_1 \overset{1}{p} ; \ldots X_1 \overset{\varrho+1}{p} \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ X_p \overset{1}{p} ; \ldots X_p \overset{\varrho+1}{p} \end{matrix} \right| . = \text{Matrix vom Range } \varrho ; \qquad X_b = B_b^\mu \, \partial_\mu \quad . \quad (4)$$

ein System von Lösungen $\overset{1}{p}, \ldots, \overset{\varrho+1}{p}$, das im Punkte $x^\varkappa = \underset{0}{x}{}^\varkappa$ der Gleichung

$$(\partial_{[\lambda_1} \overset{1}{p}) \, (\partial_{\lambda_2} \overset{2}{p}) \ldots . \, (\partial_{\lambda_{\varrho+1}]} \overset{\varrho+1}{p}) = Q_{[\lambda_1} P_{\lambda_2 \ldots . \, \lambda_{\varrho+1}]} \quad . \quad . \quad . \quad . \quad (5)$$

genügt. In dem Ausdrucke rechts ist Q_λ ein Vektor in $x^\varkappa = \underset{0}{x}{}^\varkappa$ der Teiler von $w_{\lambda_1 \ldots \lambda_q} = C_{[\lambda_1}^{p+1} \ldots C_{\lambda_q]}^n$ ist, d. h. die Gleichung

$$B_a^\lambda \, Q_\lambda = 0; \; a = 1, \ldots, p \quad . \quad . \quad . \quad . \quad . \quad (6)$$

erfüllt und der ausserdem der algebraischen Bedingung

$$Q_{[a_1 b_1} \ldots . Q_{a_\varrho b_\varrho]} \neq 0 \; ; \; Q_{ba} = 2 B^{\omega}_{[b} (\partial_{|\omega|} B^{\lambda}_{a]}) Q_\lambda \quad . \quad . \quad . \quad (7)$$

genügt, im übrigen aber beliebig wählbar ist, während $P_{\lambda_1 \ldots \lambda_\varrho}$ ein einfacher ϱ-Vektor in $x^\varkappa = \underset{0}{x^\varkappa}$ ist, der keinen Teiler von $w_{\lambda_1 \ldots \lambda_q}$ als Teiler enthält, d. h. der Ungleichung

$$v^{\varkappa_1 \ldots \varkappa_p} P_{\varkappa_1 \ldots \varkappa_\varrho} \neq 0 \; ; \; v^{\varkappa_1 \ldots \varkappa_p} = B_1^{[\varkappa_1} \ldots . B_p^{\varkappa_p]} \quad . \quad . \quad . \quad . \quad (8)$$

genügt, und der ausserdem für $p > 2$ noch den algebraischen Bedingungen

$$(\partial_{\lambda_p} v^{\lambda_1 \ldots \lambda_p}) Q_{\lambda_1} P_{\lambda_2, \ldots \lambda_{\varrho+1}} = 0 . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

unterworfen ist, im übrigen aber ebenfalls beliebig wählbar ist.

b. Es gibt mindestens einen Teiler w_λ von $w_{\lambda_1 \ldots \lambda_q}$, dessen Klasse gleich $2\varrho+1$ ist und der in $x^\varkappa = \underset{0}{x^\varkappa}$ mit Q_λ zusammenfällt.

c. Es gibt q linear unabhängige Teiler von $w_{\lambda_1 \ldots \lambda_q}$, deren Klassen höchstens gleich $2\varrho+1$ sind, dagegen gibt es keine q linear unabhängigen Teiler, deren Klassen alle kleiner als $2\varrho+1$ wären.

In dieser Mitteilung beweisen wir den Teil (*a*) des Haupttheorems für den Fall wo der Halbrang ϱ den grösst möglichen Wert $\varrho = u-1 = \left[\frac{p}{2}\right]$ hat.

In der nächsten Mitteilung folgt dann der Beweis des Teiles (*a*) für den hier noch ausgeschlossenen Fall mit kleinerem ϱ und der Teile (*b*) und (*c*).

2. *Beweis von* (a) *für maximales* $\varrho = u-1$ *und ungerades* p, $p = 2u-1$. Wir nehmen an, der Beweis sei schon geliefert für das vorhergehende gerade p, $p = 2u-2$, $\varrho = u-1$. Zunächst wählen wir die X_b so, dass

$$X_b = \partial_b + B_b^{p+1} \partial_{p+1} + \ldots . + B_b^n \partial_n \; ^1) . \quad . \quad . \quad . \quad . \quad (10)$$

ist. Infolge (7) lässt sich durch Vertauschung der Indizes $1, \ldots, p$ immer erreichen dass

$$Q_{[23} \ldots . Q_{2u-2\, 2u-1]} \neq 0 \; ; \; Q_{ba} = 2 B^{\omega}_{[b} (\partial_{|\omega|} B^{\lambda}_{a]}) Q_\lambda \quad . \quad . \quad . \quad (11)$$

ist. Ferner sind die Bedingungen (8) und (9) infolge (6) gleichbedeutend mit

$$P_{b_2, \ldots b_u} \neq 0 \; ; \; P_{b_2, \ldots b_u} = B^{\lambda_2}_{b_2} \ldots . B^{\lambda_u}_{b_u} P_{\lambda_2, \ldots \lambda_u} \; ; \; b_2, \ldots, b_u = 1, \ldots, p . \quad (12)$$

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[1]) Wir vereinfachen also die Bezeichnungsweise, indem wir sofort das speziell gewählte System der X mit X_b bezeichnen und nicht mit $X_{\mathfrak{b}}$. Vergl. die beiden vorigen Mitteilungen dieser Serie, Proc. Kon. Ned. Akad. v. Wet. **43**, 18—31, 179—188, (1940), hier als I und II zitiert.

bzw.

$$Q_{[ab}\, P_{b_2 \ldots b_u]} = 0 \;;\; a, b, b_2, \ldots, b_u = 1, \ldots, \mathrm{p} \quad . \quad . \quad . \quad (13)$$

Nehmen wir nun einmal an, dass alle Bestimmungszahlen von $P_{b_2 \ldots b_u}$, die keinen Index 1 enthalten, Null wären. Dann enthielte $P_{b_2 \ldots b_u}$ einen Faktor π_b, dessen Bestimmungszahlen $\pi_2, \ldots, \pi_\mathrm{p}$ alle Null wären.

(13) liesse sich also in der Form schreiben

$$Q_{[ab}\, \pi_{b_2}\, \pi_{b_3 \ldots b_u]} = 0 \;;\; a, b, b_2, b_3, \ldots . b_u = 1, \ldots, \mathrm{p} \quad . \quad . \quad (14)$$

wo $\pi_{b_3 \ldots b_u}$ ein einfacher $(u-2)$-Vektor wäre. Aus (14) würde aber folgen

$$Q_{[ab}\, \pi_{b_3 \ldots b_u]} = 0 \;;\; a, b, b_3, \ldots, b_u = 2, \ldots, \mathrm{p} \text{ (also nicht 1!)} \quad . \quad (15)$$

In der E_{2u-2} der Indizes $2, \ldots, \mathrm{p}$ ist nun aber Q_{ab} infolge (11) ein Bivektor vom höchsten Range $2u-2$ und aus (15) folgt also dass $\pi_{b_3 \ldots b_u} = 0$; $b_3, \ldots, b_u = 2, \ldots, \mathrm{p}$; ist [1]). Es würde also $P_{b_2 \ldots b_u} = \pi_{[b_2}\, \pi_{b_3 \ldots b_u]}$; $b_2, \ldots, b_u = 1, \ldots \mathrm{p}$; verschwinden, was jedoch im Widerspruch wäre mit (12), und daraus geht hervor, dass $P_{b_2 \ldots b_u}$ mindestens eine nicht verschwindende Bestimmungszahl hat ohne Index 1. Durch Vertauschung der Indizes $2, \ldots, \mathrm{p}$ (welche Vertauschung keinen Einfluss auf (11) hat) kann man also stets erreichen dass

$$P_{2 \ldots \mathrm{u}} \neq 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

ist.

Im $(n-1)$-dimensionalen Raum $x^1 = \underset{0}{x^1}$ mit den Koordinaten x^α; $\alpha, \beta = 2, \ldots, n$; betrachten wir jetzt die Komponenten Q_{β} und $P_{\beta_2 \ldots \beta_u}$; $\beta, \beta_2, \ldots, \beta_u = 2, \ldots, n$; sowie

$$\overset{0}{B}{}^{\alpha}_{b}\,(x^2, \ldots, x^n) = B^{\alpha}_{b}\,(\underset{0}{x^1}, x^2, \ldots, x^n) \;;\; b = 2, \ldots, \mathrm{p} \quad . \quad . \quad (17)$$

$$\overset{0}{C}{}^{x}_{\beta}\,(x^2, \ldots, x^n) = C^{x}_{\beta}\,(\underset{0}{x^1}, x^2, \ldots, x^n) \;;\; x = \mathrm{p}+1, \ldots, \mathrm{n} \quad . \quad . \quad (18)$$

$$\alpha, \beta = 2, \ldots, n$$

Da die Bestimmungszahlen B^1_b für $b \geqq 2$ infolge (10) verschwinden, ist

$$\overset{0}{C}{}^{x}_{\beta}\; \overset{0}{B}{}^{\beta}_{b} = 0 \;;\; x = \mathrm{p}+1, \ldots, \mathrm{n} \;;\; b = 2, \ldots, \mathrm{p}\;\; \beta = 2, \ldots, n. \quad (19)$$

[1]) Wir verwenden hier den Hilfssatz: Ist $Q_{\mu\lambda}$ ein Bivektor vom höchsten Range in einer E_n mit geradem $n = 2\,\mathrm{m}$ und ist $t_{\lambda_1 \ldots \lambda_{m-1}}$ ein einfacher $(m-1)$-Vektor, so folgt aus $Q_{[\mu\lambda}\, t_{\lambda_1 \ldots \lambda_{m-1}]} = 0$ das verschwinden von $t_{\lambda_1 \ldots \lambda_{m-1}}$. Der Beweis ist sehr einfach und ergibt sich durch geschickte Wahl der Massvektoren.

Im Raume $x^1 = \overset{0}{x}{}^1$ sind also die q Gleichungen

$$\overset{0}{C}{}^{x}_{\beta}\, dx^{\beta} = 0 \; ; \; x = \mathrm{p}+1, \ldots, \mathrm{n} \; ; \; \beta = 2, \ldots, n \; . \quad . \quad . \quad (20)$$

und die $p-1 = 2u-2$ Gleichungen

$$\overset{0}{B}{}^{\beta}_{b}\, \partial_{\beta} f = 0 \; ; \; b = 2, \ldots, \mathrm{p} \; ; \; \beta = 2, \ldots, n \; . \quad . \quad . \quad . \quad (21)$$

adjungierte Systeme.

Da für $x^{\alpha} = \overset{0}{x}{}^{\alpha}$ die Vektoren B_b und $\overset{0}{B}{}^{\alpha}_{b}$ für $b \geqq 2$, sowie $2B^{\omega}_{[b}\, \partial_{|\omega|}\, B^{\alpha}_{a]}$ und $2\overset{0}{B}{}^{\beta}_{[b}\, \partial_{|\beta|}\, \overset{0}{B}{}^{\alpha}_{a]}$; $b, a \geqq 2$; einander gleich sind, folgt aus (6)

$$\overset{0}{B}{}^{\beta}_{b}\, Q_{\beta} = 0 \; ; \; b = 2, \ldots, \mathrm{p} \; ; \; \beta = 2, \ldots, n \; . \quad . \quad . \quad . \quad (22)$$

im Punkte $x^{\alpha} = \overset{0}{x}{}^{\alpha}$; und aus (11), wegen des Verschwindens der ersten p Bestimmungszahlen aller $2B^{\omega}_{[b}\, \partial_{|\omega|}\, B^{\varkappa}_{a]}$,

$$Q_{[23 \ldots} \, Q_{2\mathrm{u}-2\,2\mathrm{u}-1]} \neq 0 \; ; \; Q_{ba} = 2\overset{0}{B}{}^{\beta}_{[b}\, (\partial_{|\beta|}\, \overset{0}{B}{}^{\alpha}_{a]})\, Q_{\alpha} \quad . \quad . \quad (23)$$

im selben Punkte. Aus (16) und (13) erhält man schliesslich

$$P_{2\ldots\mathrm{u}} \neq 0 \; ; \; P_{b_2 \ldots b_u} = \overset{0}{B}{}^{\beta_2}_{b_2} \ldots . \, \overset{0}{B}{}^{\beta_u}_{b_u}\, P_{\beta_2 \ldots \beta_u} \; ; \; b_2, \ldots b_u = 2, \ldots, \mathrm{p} \quad . \quad (24)$$

und

$$Q_{[ab}\, P_{b_2 \ldots b_u]} = 0 \; ; \; a, b, b_2, \ldots, b_u = 2, \ldots, \mathrm{p} \quad . \quad . \quad . \quad (25)$$

für $x^{\alpha} = \overset{0}{x}{}^{\alpha}$.

Die Grössen Q_{β} und $P_{\beta_2 \ldots \beta_u}$ genügen also, infolge (22), (23), (24) und (25), den Bedingungen (6), (7), (8) und (9) für das im Raume $x^1 = \overset{0}{x}{}^1$ liegende PFAFFsche System (20). Da nun aber angenommen war, dass Teil (*a*) des Haupttheorems schon für $p = 2u-2$ und $\varrho = u-1$ bewiesen ist, existieren somit u in der Umgebung des Punktes $x^{\alpha} = \overset{0}{x}{}^{\alpha}$ analytische Funktionen $\overset{1}{p}_0, \ldots, \overset{u}{p}_0$ die den Gleichungen

$$\begin{vmatrix} \overset{0}{X}_2\, \overset{1}{p}_0, & \ldots, & \overset{0}{X}_2\, \overset{u}{p}_0 \\ \vdots & & \vdots \\ \overset{0}{X}_{\mathrm{p}}\, \overset{1}{p}_0, & \ldots, & \overset{0}{X}_{\mathrm{p}}\, \overset{u}{p}_0 \end{vmatrix} = \text{Matrix vom Range} < u \quad . \quad . \quad . \quad (26)$$

$$\overset{0}{X}_b = \overset{0}{B}{}^{\beta}_{b}\, \partial_{\beta} \; ; \; b = 2, \ldots, \mathrm{p} \; ; \; \beta = 2, \ldots, n$$

und in $x^\alpha = \underset{0}{x^\alpha}$ den Bedingungen

$$(\partial_{[\beta_1} \overset{1}{p_0}) \ldots . (\partial_{\beta_u]} \overset{u}{p_0}) = Q_{[\beta_1} P_{\beta_2 \ldots \beta_u]} \quad . \quad . \quad . \quad . \quad . \quad (27)$$

genügen.

Von diesen Funktionen beweisen wir nun, dass sie den Bedingungen (II 12), (II 13) und (II 14) [1]) genügen. Aus (27) folgt in $x^\alpha = \underset{0}{x^\alpha}$

$$\overset{0}{B}{}_2^{\beta_2} \ldots . \overset{0}{B}{}_u^{\beta_u} (\partial_{[\beta_1} \overset{1}{p_0}) \ldots . (\partial_{\beta_u]} \overset{u}{p_0}) = \frac{1}{u} Q_{\beta_1} P_{2 \ldots u} \quad . \quad . \quad . \quad (28)$$

Da Q_{β} wegen (23) nicht Null sein kann und auch $P_{2 \ldots u}$ infolge (24) nicht verschwinden kann, ist die linke Seite von (28) nicht Null. Daraus geht hervor, dass (II 12) erfüllt ist. Ferner lassen sich die Determinanten der Matrix (26), die die ersten $u-1$ Reihen enthalten, schreiben

$$(\overset{0}{X}_{[k} \overset{1}{p_0}) (\overset{0}{X}_2 \overset{2}{p_0}) \ldots \overset{0}{X}_{u]} \overset{u}{p_0}) = (X_{[k} \overset{1}{p_0})_0 (X_2 \overset{2}{p_0})_0 \ldots (X_{u]} \overset{u}{p_0})_0 ; \; k = u+1, \ldots, p \quad (29)$$

und aus (26) folgt also dass auch (II 14) erfüllt ist. Aus der Definition (II 8*b*) von Ψ_{ik}, (22) und (27) folgt des weiteren

$$\left.\begin{aligned} \Psi_{ik} &= \frac{1}{u} \{ Q_{ik} P_{2 \ldots u} - Q_{i2} P_{k3 \ldots u} + \ldots . + (-1)^u Q_{iu} P_{k2 \ldots u-1} \} = \\ &= Q_{i[k} P_{2 \ldots u]} \end{aligned}\right\} \quad (30)$$

in $x^z = \underset{0}{x^z}$, sodass (II 13) in $x^z = \underset{0}{x^z}$ geschrieben werden kann

$$\triangle = \begin{vmatrix} Q_{2[u+1} P_{2 \ldots u]} & \ldots . & Q_{u[u+1} P_{2 \ldots u]} \\ \vdots & & \vdots \\ Q_{2[p} P_{2 \ldots u]} & \ldots . & Q_{u[p} P_{2 \ldots u]} \end{vmatrix} \neq 0 \quad . \quad . \quad . \quad (31)$$

Nun können wir aber für $\triangle^2$ schreiben

$$\triangle^2 = u^{2-2u} \begin{vmatrix} Q_{22} P_{2 \ldots u} & \ldots Q_{u2} P_{2 \ldots u} & -u Q_{2[u+1} P_{2 \ldots u]} & \ldots & -u Q_{2[p} P_{2 \ldots u]} \\ \vdots & \vdots & \vdots & & \vdots \\ Q_{2u} P_{2 \ldots u} & \ldots Q_{uu} P_{2 \ldots u} & -u Q_{u[u+1} P_{2 \ldots u]} & \ldots & -u Q_{u[p} P_{2 \ldots u]} \\ u Q_{2[u+1} P_{2 \ldots u]} & \ldots u Q_{u[u+1} P_{2 \ldots u]} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ u Q_{2[p} P_{2 \ldots u]} & \ldots u Q_{u[p} P_{2 \ldots u]} & 0 & \ldots & 0 \end{vmatrix} \quad (32)$$

[1]) Beiträge zur Theorie der Systeme PFAFFscher Gleichungen, II, Proc. Kon. Ned. Akad. v. Wet. **43**, 179—188, (1940).

und durch Umrechnung dieser Determinante ergibt sich

$$\triangle^2 = u^{2-2u}\,(P_{2\ldots u})^{2u-2}\begin{vmatrix} Q_{22} & \ldots & Q_{p2} \\ \vdots & & \\ Q_{2p} & \ldots & Q_{pp} \end{vmatrix} \quad . \quad . \quad . \quad . \quad (33)$$

oder

$$\triangle^2 = u^{2-2u}\left(\frac{(2u-2)!}{(u-1)!}\right)^2 (P_{2\ldots u})^{2u-2}\,(Q_{[23}\ldots . Q_{p-1\,p]})^2 \quad . \quad . \quad (34)$$

Infolge (11) und (16) ist also $\triangle \neq 0$ in $x^z = \overset{0}{x}{}^z$, also auch in einer Umgebung von diesem Punkt und es ist also auch die Bedingung (II 13) erfüllt. In (II) wurde aber bewiesen, dass es dann auch Funktionen $\overset{1}{p},\ldots,\overset{u}{p}$ von $x^1,\ldots,x^n$ gibt, die den Gleichungen

$$\begin{vmatrix} X_1\overset{1}{p} & \ldots & X_1\overset{u}{p} \\ \vdots & & \vdots \\ X_p\overset{1}{p} & \ldots & X_p\overset{u}{p} \end{vmatrix} = \text{Matrix vom Range} < u \quad . \quad . \quad . \quad (35)$$

genügen und für $x^1 = \overset{0}{x}{}^1$ übergehen in $\overset{1}{p}_0,\ldots,\overset{u}{p}_0$. Es ist also nur noch zu beweisen, dass diese Funktionen in $x^z = \overset{0}{x}{}^z$ auch den Gleichungen

$$\partial_{[\lambda_1}\overset{1}{p}\ldots.\,\partial_{\lambda_u]}\overset{u}{p} = Q_{[\lambda_1 P_{\lambda_2\ldots\lambda_u]}} \;;\; \lambda_1,\ldots,\lambda_u = 1,\ldots,n \quad . \quad . \quad (36)$$

genügen. Jedenfalls folgt schon aus (27), dass in $x^z = \overset{0}{x}{}^z$

$$(\partial_{[\beta_1}\overset{1}{p})\ldots.\,(\partial_{\beta_u]}\overset{u}{p}) = Q_{[\beta_1 P_{\beta_2\ldots\beta_u]}} \;;\; \beta_1,\ldots,\beta_u = 2,\ldots,n \quad . \quad (37)$$

ist. Da $Q_{[\lambda_1 P_{\lambda_2\ldots\lambda_u]}}$ einfach ist, kann man schreiben

$$Q_{[\lambda_1 P_{\lambda_2\ldots\lambda_u]}} = \overset{1}{P}_{[\lambda_1}\ldots.\,\overset{u}{P}_{\lambda_u]} \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

und infolge (37) lassen sich die Faktoren so wählen, dass in $x^z = \overset{0}{x}{}^z$

$$\partial_\beta\overset{1}{p} = \overset{1}{P}_\beta;\ldots;\partial_\beta\overset{u}{p} = \overset{u}{P}_\beta \;;\; \beta = 2,\ldots,n \quad . \quad . \quad . \quad . \quad (39)$$

ist. Nun lassen sich $\partial_1\overset{1}{p},\ldots,\partial_1\overset{u}{p}$ bestimmen mittels (II, 10), in welchen Gleichungen für $\overset{2}{\sigma},\ldots,\overset{u}{\sigma}$ die Werte eingesetzt werden müssen, die man erhält indem man ζ_k (II 8a) gleich Null setzt. Diese Lösungen existieren,

da jetzt tatsächlich (II 9) erfüllt ist. Wir schreiben jetzt (II 10) (identisch mit (II 6a)) in der Form

$$\left.\begin{aligned} \overset{1}{p}_1 &= \overset{2}{\sigma}\,\overset{1}{p}_2 + \ldots + \overset{u}{\sigma}\,\overset{1}{p}_u \\ &\vdots \\ \overset{u}{p}_1 &= \overset{2}{\sigma}\,\overset{u}{p}_2 + \ldots + \overset{u}{\sigma}\,\overset{u}{p}_u \end{aligned}\right\} \quad p_i = B_i^\mu\, \partial_\mu p \;;\; i = 1, \ldots, u \quad . \quad . \quad (40)$$

und bemerken, dass die Gleichungen zur Bestimmung von $\overset{2}{\sigma}, \ldots, \overset{u}{\sigma}$ für $x^\varkappa = \overset{\varkappa}{x}_0$ wegen (30) übergehen in

$$\zeta_k = Q_{1[k}\, P_{2\ldots u]} - \overset{2}{\sigma}\, Q_{2[k}\, P_{2\ldots u]} - \ldots - \overset{u}{\sigma}\, Q_{u[k}\, P_{2\ldots u]} = 0 \;;\; k = u+1, \ldots, p \quad (41)$$

Anderseits ist, wegen des Verschwindens von $B_b^\mu\, Q_\mu$ und (38)

$$\overset{1}{P}_{[1} \ldots . \overset{u}{P}_{u]} = 0 \;;\; P_i = B_i^\mu\, P_\mu \;;\; i = 1, \ldots, u \quad . \quad . \quad . \quad (42)$$

und es müssen also zwischen den P_i Gleichungen bestehen von der Form

$$\left.\begin{aligned} \overset{1}{P}_1 &= \overset{2}{\tau}\,\overset{1}{P}_2 + \ldots + \overset{u}{\tau}\,\overset{1}{P}_u \\ &\vdots \\ \overset{u}{P}_1 &= \overset{2}{\tau}\,\overset{u}{P}_2 + \ldots + \overset{u}{\tau}\,\overset{u}{P}_u \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Es ist also nur noch zu beweisen, dass die τ in (43) mit den σ in (40) identisch sind. (Aus (39) geht hervor, dass $\overset{i}{P}_k = \overset{i}{p}_k$ ist für $i = 1, \ldots, u$; $k = 2, \ldots, u$).

Dazu gehen wir aus von (13) und folgern daraus

$$Q_{[kb}\, P_{2\ldots u]} = 0 \;;\; k = u+1, \ldots, p \;;\; b = 1, \ldots, p \quad . \quad . \quad . \quad (44)$$

welche Gleichung sich folgendermassen ausschreiben lässt

$$2\, Q_{b[k}\, P_{2\ldots u]} + (-1)^u\, (u-1)\, Q_{[k2}\, P_{3\ldots u]b} = 0 \quad . \quad . \quad . \quad (45)$$

Infolge (38) und (6) ist diese Gleichung aber gleichwertig mit

$$\left.\begin{aligned} Q_{b[k}\, P_{2\ldots u]} &= (-1)^{u-1}(u-1)(B_{[k}^\omega\, \partial_{|\omega|}\, B_2^{\lambda_2}) B_3^{\lambda_3} \ldots . B_{u]}^{\lambda_u}\, B_b^\lambda\, Q_{\lambda_2 P_{\lambda_3 \ldots . \lambda_u \lambda}} = \\ &= (-1)^{u-1}(u-1)\, u\,!\,(B_{[k}^\omega\, \partial_{,\omega|}\, B_2^{\lambda_2}) B_3^{\lambda_3} \ldots B_{u]}^{\lambda_u}\, B_b^\lambda\, \overset{[1}{P}_{\lambda_2} \ldots \overset{u-1}{P}_{\lambda_u}\, \overset{u]}{P}_\lambda = \\ &= (-1)^{u-1}(u-1)\, u\,!\,(B_{[k}^\omega\, \partial_{|\omega|}\, B_2^\lambda)\, \overset{[1}{P}_{|\lambda|}\, \overset{2}{P}_3 \ldots . \overset{u-1}{P}_{u]}\, \overset{u]}{P}_b \end{aligned}\right\} . \quad (46)$$

woraus hervorgeht, dass

$$\left.\begin{aligned} &Q_{1[k}\, P_{2\ldots u]} - \overset{2}{\tau}\, Q_{2[k}\, P_{2\ldots u]} - \ldots . - \overset{u}{\tau}\, Q_{u[k}\, P_{2\ldots u]} = \\ &= (-1)^{u-1}(u-1)\, u\,!\,(B_{[k}^\omega\, \partial_{|\omega|}\, B_2^\lambda)\, \overset{[1}{P}_{|\lambda|}\, \overset{2}{P}_3 \ldots \overset{u-1}{P}_{u]} (\overset{u]}{P}_1 - \overset{|2|\,u]}{\tau}\, P_2 - \ldots - \overset{|u|\,u]}{\tau}\, P_u) = 0 \end{aligned}\right\} \quad (47)$$

ist, womit dargetan ist, dass $\overset{2}{\tau}, \ldots, \overset{u}{\tau}$ ebenfalls Lösungen sind der Gleichungen (41), deren Determinante $\triangle$ infolge (34) nicht verschwindet, sodass die τ mit den σ zusammenfallen:

$$\overset{1}{p}_1 = \overset{1}{P}_1 ; \ldots . ; \overset{u}{p}_1 = \overset{u}{P}_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

Ausgeschrieben lauten diese Gleichungen

$$(\partial_1 \overset{i}{p} - \overset{i}{P}_1) + B_1^{p+1}(\partial_{p+1} \overset{i}{p} - \overset{i}{P}_{p+1}) + \ldots + B_1^n (\partial_n \overset{i}{p} - \overset{i}{P}_n) = 0 ; \; i = 1, \ldots, u \; . \quad (49)$$

und aus (39) folgt also in $x^\varkappa = \underset{0}{x^\varkappa}$

$$\partial_1 \overset{1}{p} = \overset{1}{P}_1 ; \ldots ; \partial_1 \overset{u}{p} = \overset{u}{P}_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

also, kombiniert mit (39)

$$\partial_\lambda \overset{1}{p} = \overset{1}{P}_\lambda ; \ldots ; \; \partial_\lambda \overset{u}{p} = \overset{u}{P}_\lambda ; \; \lambda = 1, \ldots, n \quad . \quad . \quad . \quad . \quad (51)$$

woraus, unter Berücksichtigung von (38) die zu beweisende Gleichung (36) folgt. Die Existenz der Lösungen $\overset{1}{p}, \ldots, \overset{u}{p}$ von (4), die in $x^\varkappa = \underset{0}{x^\varkappa}$ der Bedingung (5) genügen, ist damit für ungerades p und $\varrho = \frac{1}{2}(p-1)$ bewiesen. Da ausserdem die linke Seite von (28), wie bemerkt, nicht verschwinden kann, ist

$$(X_2 \overset{[i_2}{p}) \ldots (X_u \overset{i_u]}{p}) \neq 0 \; ; \; i_2, \ldots, i_u = 1, \ldots, u \; . \quad . \quad . \quad . \quad (52)$$

und hat (35) demnach genau den Rang $u-1$. Der Beweis ist damit für diesen Fall vollständig erledigt.

3. *Beweis von* (a) *für maximales* $\varrho = u-1$ *und gerades* p, $p = 2u-2$. Wir nehmen an, der Beweis sei schon geliefert für das vorhergehende ungerade p, $p = 2u-3$ und maximales $\varrho = u-2$. Infolge (8) besitzt $P_{\lambda_2 \ldots \lambda_u}$ einen Teiler P_λ, für welchen in $x^\varkappa = \underset{0}{x^\varkappa}$ gilt

$$B_b^\lambda P_\lambda \neq 0 \; ; \; b = 1, \ldots, p \; . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

Es gibt also wenigstens einen Vektor, z.B. $B_1^\varkappa$, so dass in $x^\varkappa = \underset{0}{x^\varkappa}$

$$B_1^\lambda P_\lambda \neq 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

ist. Wir transformieren jetzt das Koordinatensystem, so dass für das neue System (das wir aber einfachheitshalber wieder mit $x^\varkappa$ bezeichnen) überall

$$B_1^\varkappa = \underset{1}{e^\varkappa} \; . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

ist und in $x^z = \underset{0}{x^z}$

$$P_\lambda = \overset{1}{e}_\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

Durch lineare Transformationen der B'_b bilden wir ferner die B'_b in solcher Weise, dass

$$B^1_b = 0 \; ; \; b = 2, \ldots, \mathrm{p} \text{ (also nicht 1!)} \quad . \quad . \quad . \quad . \quad . \quad (57)$$

ist. Infolge (56) lässt sich dann $P_{\lambda_2 \ldots \lambda_u}$ schreiben

$$P_{\lambda_2 \ldots \lambda_u} = \overset{1}{e}_{[\lambda_2} P'_{\lambda_3 \ldots \lambda_u]} \quad , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

wo $P'_{\lambda_3 \ldots \lambda_u}$ ein einfacher $(u-2)$-Vektor ist. Da aber $P_{b_2 \ldots b_u} = B^{\lambda_2}_{b_2} \ldots B^{\lambda_u}_{b_u} P_{\lambda_2 \ldots \lambda_u}$; $b_2, \ldots, b_u = 1, \ldots \mathrm{p}$ infolge (8) nicht verschwindet, folgt aus (55), (57) und (58)

$$P'_{b_3 \ldots b_u} \neq 0 \; ; \; b_3, \ldots, b_u = 2, \ldots, \mathrm{p} \quad . \quad . \quad . \quad . \quad . \quad (59)$$

Nun ist (9) gleichwertig mit

$$Q_{[ab} P_{b_2 \ldots b_u]} = 0 \; ; \; a, b, b_2, \ldots, b_u = 1, \ldots, \mathrm{p} \quad . \quad . \quad . \quad (60)$$

und unter Berücksichtigung von (55), (57) und (58) folgt daraus

$$Q_{[ab} P'_{b_3 \ldots b_u]} = 0 \; ; \; a, b, b_3, \ldots, b_u = 2, \ldots, \mathrm{p} \quad . \quad . \quad . \quad (61)$$

Infolge (7) ist ferner

$$Q_{[1\,2} \ldots . \, Q_{2u-3\;2u-2]} \neq 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

und daraus folgt, dass nicht *alle* Alternationen über $2u-4$ der Indizes von $2, \ldots, \mathrm{p}$ von $u-2$ Faktoren Q_{ab} verschwinden können. D.h. es ist

$$Q_{[a_3 b_3} \ldots Q_{a_u b_u]} \neq 0 \; ; \; a_3, \ldots, a_u, b_3, \ldots, b_u = 2, \ldots, \mathrm{p} \quad . \quad . \quad (63)$$

Wir betrachten jetzt den einfachen $(p-1)$-Vektor $B^{[\varrho}_2 \ldots B^{\varrho p]}_{\mathrm{p}}$ und den zugehörigen einfachen $(q+1)$-Vektor. In bezug auf das zu diesem $(q+1)$-Vektor gehörige System von $q+1$ PFAFFschen Gleichungen genügen Q_λ und $P'_{\lambda_3 \ldots \lambda_u}$ den Bedingungen (6), (7) (wegen (63)), (8) (wegen (59)) und (9) (wegen 61)). Da wir bei diesem System gerade den Fall vor uns haben mit dem vorhergehenden ungeraden p-Wert ($p-1 = 2u-3$ und $\varrho = u-2$) existiert ein System von $u-1$ Funktionen $\overset{2}{p}, \ldots, \overset{u}{p}$ von $x^1, \ldots, x^n$, die den Gleichungen

$$\left\| \begin{array}{ccc} X_2 \overset{2}{p} & \ldots . & X_2 \overset{u}{p} \\ \vdots & & \vdots \\ X_{\mathrm{p}} \overset{2}{p} & \ldots . & X_{\mathrm{p}} \overset{u}{p} \end{array} \right\| = \text{Matrix vom Range } u-2. \quad . \quad . \quad (64)$$

und ausserdem in $x^\varkappa = \underset{0}{x}^\varkappa$ den Bedingungen

$$(\partial_{[\lambda_2} \overset{2}{p}) \ldots . (\partial_{\lambda_u]} \overset{u}{p}) = Q_{[\lambda_2} P'_{\lambda_3 \ldots \lambda_u]} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

genügen. Wählt man nun $\overset{1}{p} = -x^1$, so ist

$$\left\| \begin{matrix} X_1 \overset{1}{p} & \ldots . & X_1 \overset{u}{p} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ X_p \overset{1}{p} & \ldots . & X_p \overset{u}{p} \end{matrix} \right\| = \left\| \begin{matrix} -1 & 0 & \ldots . & 0 \\ 0 & X_2 \overset{2}{p} & \ldots . & X_2 \overset{u}{p} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & X_p \overset{2}{p} & \ldots . & X_p \overset{u}{p} \end{matrix} \right\| = \text{Matrix vom Range } u-1 \quad (66)$$

und ausserdem ist in $x^\varkappa = \underset{0}{x}^\varkappa$ infolge (65)

$$(\partial_{[\lambda_1} \overset{1}{p}) \ldots (\partial_{\lambda_u]} \overset{u}{p}) = -\overset{1}{e}_{[\lambda_1} Q_{\lambda_2} P'_{\lambda_3 \ldots \lambda_u]} = Q_{[\lambda_1} P_{\lambda_2 \ldots \lambda_u]} \cdot \quad . \quad . \quad . \quad (67)$$

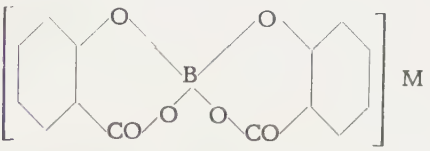
sodass der Beweis jetzt auch für gerades p und $\varrho = {}^1/_2 p$ erledigt ist.

Chemistry. — *The behaviour of the ortho-hydroxy-naphtoic acids 2.3, 2.1 and 1.2 in respect to a 0.5 molar aqueous boracic acid solution.* By J. BÖESEKEN and A. NIKS.

(Communicated at the meeting of March 30, 1940.)

§ 1. It was previously demonstrated that there are a number of ortho-hydroxy benzene carbonic acids which by boracic acid in an aqueous solution, exhibit a very considerable increase in conductivity [1]).

Also salts of the type

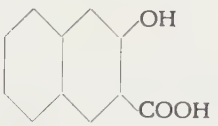


and similar ones of bi-valent kations were readily obtained. MEULENHOFF, by splitting the *strychnine salt* into optic antipodes, succeeded in demonstrating that the borium atom stands in the centre of a tetraeder [2]).

Such an extraordinary increase in the conductivity, even with moderately concentrated solutions, indicated a stand of the hydroxyl groups highly favourable to the formation of a six-atomic ring, which may be taken as a proof of the theory that the OH and COOH groups are held at these favourable places in the plane of the rigid benzene ring.

As the naphtalene ring corresponds concerning its sterical conditions completely to the benzene ring, we expected analogous phenomena with the hydroxy naphtoic acids, with merely slight differences between the three isomers.

In the initial experiments with the 2.3 acid no increase of the conductivity was observed.



It further appeared, however, that the solubility was very poor.

Owing to this property, the solubility could not be accurately enough measured in the usual way, e.g. by weighing and titration.

We, therefore, applied the method of determining the conductivity. Very weak solutions were prepared by weighing the acids upon the micro-

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[1]) Rec. trav. Chim. **40**, 574 (1920).

[2]) Proc. Kon. Akad. v. Wetensch., Amsterdam, **26**, 32 and 97 (1923); J. MEULENHOFF, Dissertation, Delft 1924.

balance, and from this the conductivity in water at 25° was determined.

We then get a number of points which, in a graph, come to lie in an almost straight line. This line terminates at the moment that the maximum solubility is reached, because the conductivity increases no further; by taking an excess of acid, this point could be determined. It follows, moreover, that the conductivity over the whole field of solubility was found.

Since the conductivity of all three naphtol carbonic acids is very considerable, the method is sufficiently accurate.

Thereafter the same method was applied to a solution of the three acids in a 0.5 molar boracic acid solution. From the two curves it was possible by graphic interpolation to compare the conductivity of equivalent solutions in water and in boracic acid solutions, and the course of the effect of 0.5 mol. H_3BO_3 upon the conductivity. In the Tables and Diagrams given below, we have given the result of the research, and would remark here that the values obtained by interpolation for maximum solubility and for the effect of 0.5 m. H_3BO_3 upon the conductivity of the three acids are enclosed in a [ ].

Further, the conductivities given in the diagrams in 0.5 mol. H_3BO_3, are not those observed, but those minus the conductivity of 0.5 m. H_3BO_3 = 30 mohs $\times$ 10^{-6}, so that the course of the effect of this acid upon the conductivity of the hydroxy acid can be readily read.

TABLE I.

Conductivity of hydroxy 2 naphtoic acid 3.

in H_2O at 25°			in 0.5 mol. H_3BO_3 at 25°			
m.g. p. 100 cc.	gr.m.$\times 10^{-4}$ p. L.	K Kh.-Hb$\times 10^{-6}$	m.g. p. 100 cc.	gr.m.$\times 10^{-4}$ p, L.	K_1 Kh.-Hb$\times 10^{-6}$	$\triangle$ $K_1-(K+H_3BO_3)$
1.18	0.626	17.6	1.486	0.79	43.2	— 6
2.27	1.21	35.2	2.39	1.27	55.1	—11
2.97	1.59	51.6	3.77	2.00		—19
3.98	2.12	60.8	4.80	2.56		—17
[10.4]	5.53	[159.0]	[10.40]	5.53	[172.0]	—17
50	—	159.0	10.94	5.82	173.8	
75	—	160.0	[19.4]	10.32	[324.0]	
100	—	158.0	150	—	323.8	
			200	—	324.2	

As this acid exhibits the greatest difference from what was a priori expected of it, we may dwell briefly upon this result.

Firstly, the solubility is extremely poor, only 5.53 $\times$ 10^{-4} molar at 25°; in 0.5 m boracic acid it is considerably greater, viz. 10.32 $\times$ 10^{-4} mol. This is of importance, because from this it appears that a complex is

formed in spite of the fact that there is no positive effect in the entire field of solubility. Indeed, it appears that, if to a suspension of the acid in water, a little boracic acid and dimethylaniline be added, (approx. the

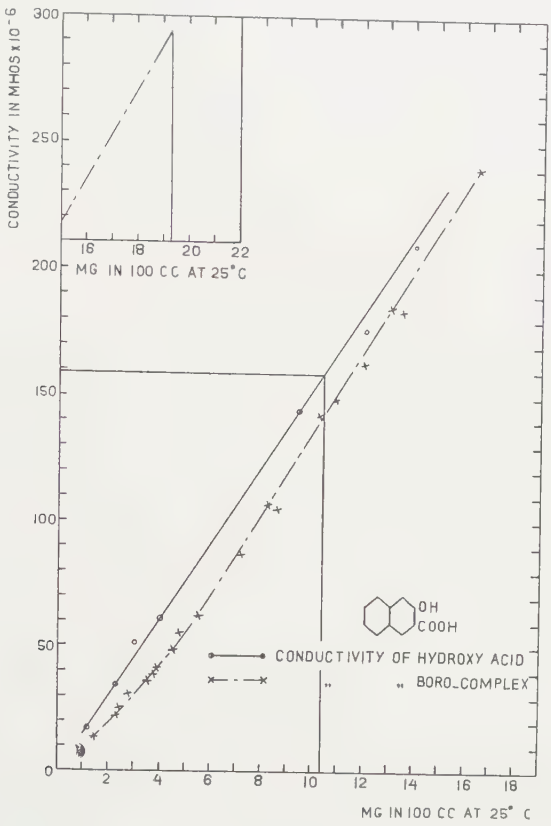


Diagram I.

molecular ratio of 2 : 1 : 1), tiny pale yellowish neadles will be seen beside the yellow crystals of the acid. By picking them out and washing by absolute alcohol, in which hydroxy acid and boracic acid dissolve readily, a little of the salt could be gathered, and, qualitatively, the presence of nitrogen and borium established.

To obtain a larger quantity, a solution of 19.4 gr. hydroxy acid, 3.31 gr. H_3BO_3 and 5.63 gr. $C_6H_5N(CH_3)_2$ was dissolved in 0.5 litre of boiling absolute alcohol; when cooled and grafted with a little of the pale yellow neadles, a large quantity of the salt was obtained (67 % of the theoretical quantum). This was dried, washed by alcohol, and dried again.

Analysis:

Found: C = 71.2 in 71.2 %; H = 4.9 and 4.9 %; N = 2.67 and 2.71 %; B = 2.62 and 2.66 %.

Calculated for $(C_{10}H_6CO_3)_2$ $BH.N(CH_3)_2C_6H_5$: C = 71.1 %; H = 4.8 %; N = 2.8; B = 2.2 %.

Although the percentage of borium was found slightly too high, there can be no doubt but that we have found a salt of the didiolboracic acid mentioned in the introduction.

The isolation of this salt does not imply that in the solutions, which have been used for determining the conductivity, didiol. acid is present in any great quantity. This is certainly *not* the case, as we shall presently explain further; we obtained it here, because this salt is highly insoluble, and because we selected the relative proportion of the components and the solvent (alcohol) so that the separation might be expected if boro-complexes occur.

TABLE II.

Conductivity of hydroxy 2 naphtoic acid 1.

in H_2O at 25°			in 0.5 H_3BO_3 at 25°			
m.g. p. 100 cc.	gr.-m.$\times 10^{-4}$ p. L.	K Kh.-Hb.$\times 10^{-6}$	m.g p. 100 cc.	gr.-m.$\times 10^{-4}$ p. L.	K_1 Kh.-Hb.$\times 10^{-6}$	$\triangle$ $K_1-(K+H_3BO_3)$
0.85	0.45	10.4	1.16	0.61[6]	39.9	— 4
2.17	1.15[5]	21.6	5.85	3.10	101.4	+ 2
3.25	1.73	37.1	10.00	5.32	164.4	+14
4.04	2.15	44.4	14.01	7.45	210	+27
14.01	7.45	173	20.0	10.64	310.1	
100	—	169	30.0	15.96	414.5	
150	—	176	40.0	21.26	523.9	
			84	44.7	1125	
			100	—	1064	
			150	—	1162	

In Table II, and the accompanying diagram, we see that although in very dilute solutions of naphtol. 2 carbonic acid 1, there is no positive effect, this is very quickly seen, and, at the maximum solubility it is even very considerable and of the same magnitude as observed with salicylic acid.

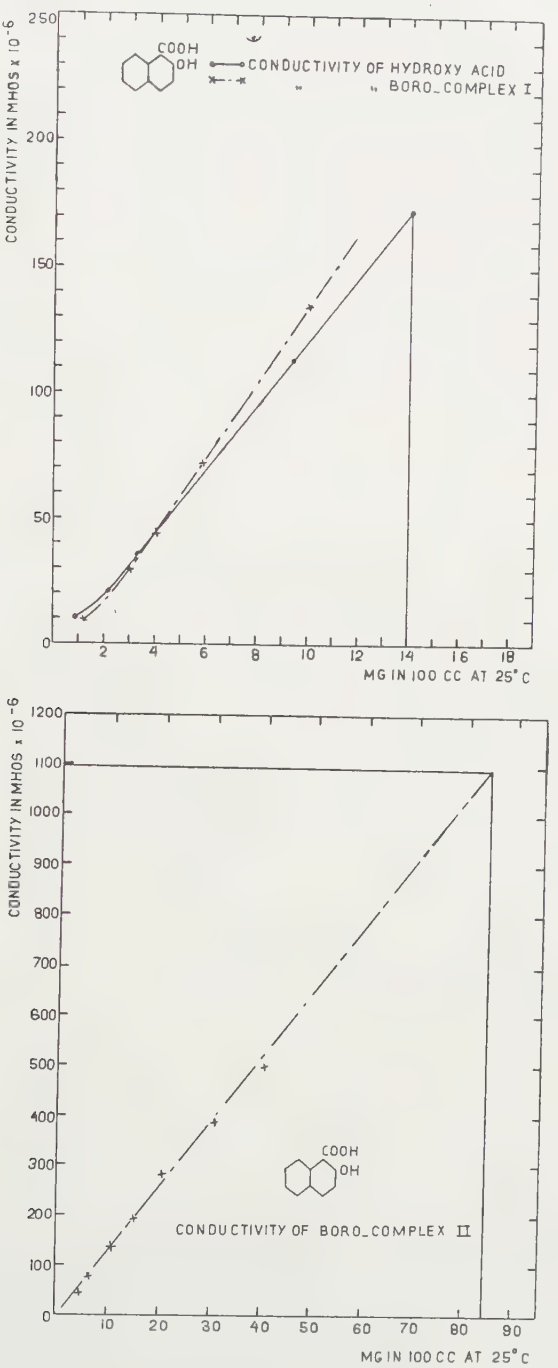
In boracic acid the solubility rises from 14 to 84 per 100 cc, i.e. to the six-fold.

In the same way as with 2.3 acid, we prepared a complex salt with dimethylaniline; only we heated the alcohol solution not more than to 40°, as at a higher temperature the hydroxy acid splits into β naphtol and CO_2.

Analysis:

Found: C = 69.6 and 69.6 %; H = 4.9 and 5.0 %; N = 2.9 and 2.8 %; B = 2.5 and 2.4 %.

Calculated for $[C_{11}H_6O_3]_2BHN(CH_3)_2C_6H_5$: C = 71.1 %; H = 4.8 %; N = 2.8 %; B = 2.2 %.



Diagrams II and IIa.

The too low C percentage shows that a slight decomposition of the salt took place; otherwise there remains no doubt as to the above composition of a didiolsalt.

TABLE III.

Conductivity of the hydroxy 1 naphtoic acid 2.

in water at 25°			in 0.5 H_3BO_3 at 25°			
m.g. p. 100 cc.	gr.m. $\times 10^{-4}$ p. L.	K Kh.-Hb $\times 10^{-6}$	m.g. p. 100 cc.	gr.m. $\times 10^{-4}$ p. L.	K_1 Kh.-Hb $\times 10^{-6}$	$\triangle$ $K_1-(K+H_3BO_3)$
1.20	0.64	13.6	1.72	0.92	42.8	— 8
1.89	1.01	22.7	3.32	1.76[6]	66.9	— 6
2.59	1.38	30.6	5.89	3.13	107.8	— 1
6.3	3.35	82.9	6.3	3.35	113.5	+ 0.6
75	—	82.9	11.63	6.20	174.6	+ 12?
100	—	82.9	18.0	9.52	296	+ 25
			75	—	297	
			100	—	296	

In Table III and the accompanying diagram we see that this acid, in its more or less abnormal behaviour, corresponds rather to the 2.3 acid

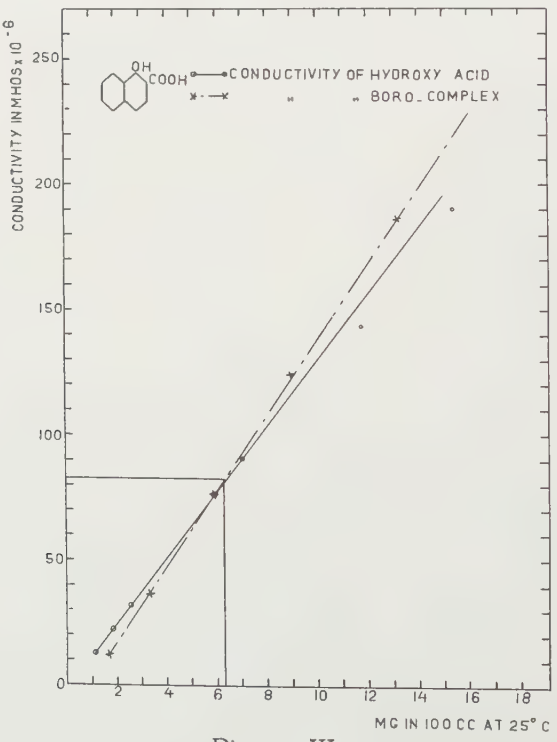


Diagram III.

than to the 2.1 acid. Nevertheless, with the maximum solubility, which is very poor here, we just reach the field of positive effect.

We did not consider it necessary to prepare a complex salt, because the solubility in 0.5 mol. H_3BO_3 being about 3 times more than that in pure water, the formation of a complex is demonstrated adequately enough in connection with the observations made with the two isomers.

§ 2. How is this negative behaviour with these hydroxy acids to be explained, since we know with certainty that derivates of the coordinative 4-valent borium are present?

In Table IV we show the solubility in water, and in 0.5 mol. H_3BO_3 at 25°.

TABLE IV.

Solubility at 25°	in H_2O m.g. p. L.	in 0.5 m. H_3BO_3 m.g. p. L.	Relative molec. con. Hydroxy acid : bor. ac.
Hydroxy 2 naphtol acid 3	104	194	$1.03 \times 10^{-3} : 0.5$
„ 2 „ „ 1	140	840	$4.47 \times 10^{-3} : 0.5$
„ 1 „ „ 2	63	180	$0.96 \times 10^{-3} : 0.5$

From the above we see that the solubility of these acids is very low, and that also in 0.5 mol. boracic acid the ratio of the concentrations is inimical to hydroxy acid.

From the researches of N. VERMAAS [3]) we know that in solutions with an excess of boracic acid and low concentrations of the diol mainly monodiolboracic acid is present. COOPS [4]), also, at an earlier date, by measuring the conductivity and applying the in aqueous solutions existing equilibria, was able to deduce that with moderate dilutions of a number of hydroxy acids in 0.5 mol. boracic acid, there must be almost exclusively monohydroxy acid boracic acid present. These dilutions have been collected in Table V.

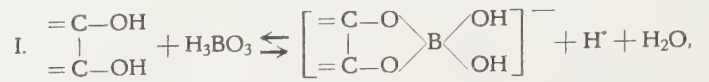
TABLE V.

Hydroxy acid	V
Glycolic acid	> 128
Amygdalic acid	> 256
Tartronic acid	> 512
d tartaric acid	> 120
anti „ „	> 120
citric acid	> 640

[3]) N. VERMAAS, Rec. trav. Chim. **51**, 67 and 955 (1932) and Dissertation, Delft 1931.
[4]) Dissertation, Delft 1924, pp. 86—88.

If we consider that the dilutions of the naphtol carbonic acid, with the maximum solubility in boracic acid, are 971, 224 and 1042 respectively, we may conclude from this that in the solutions we examined there must be almost exclusively monodiol boracic acid complexes present.

In the aqueous boracic polyol solutions, as has been demonstrated more than once, we have to do with the following equilibria:

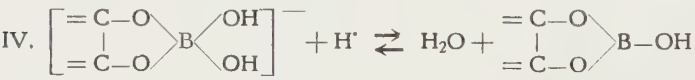


and moreover, in the case of a hydroxy acid:

III. hydroxy acid $\rightleftarrows$ hydroxy acid ion + $H^{\cdot}$

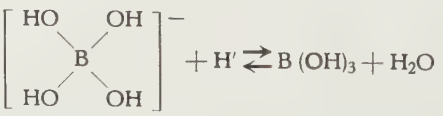
We have pointed out that the didiol boracic acids are very strong acids, to compare with to hydrochloric, nitric and similar acids, and there is no reason whatever *not* to assume this in the case of monodiol boracic acids, since they are likewise derived from the co-ordinative 4-valent borium.

However, in this case, water can be split under passing to a derivative of the 3-valent borium, and thus a non-electrolyte be formed, viz.



as is not possible with the didiol boracic acid. If, therefore, we are in the region in which mainly monodiol boric acid is present, then this possibility, which is doubtless connected with steric factors, since the distances B—O, as well as the angles are involved, may bring about a material reduction of the $H^{\cdot}$ concentration, and therewith of the conductivity.

Here we have to do with the same conditions as with boracic acid itself, which is an extremely weak acid, because the element in it is almost entirely in the 3-valent state.



By calculating the equivalent conductivity of the three naphtol carbonic acids, e.g. from the conductivity of a 0.0005 m. solution. we see that the naphtol carbonic acids are very strong acids and thus, when greatly diluted, are considerably ionised. If these acids are converted into the complex monodiol acids, then, for the above reasons, a lower, rather than a higher,

conductivity may be expected, and this naturally in the first place with the strongest acids.

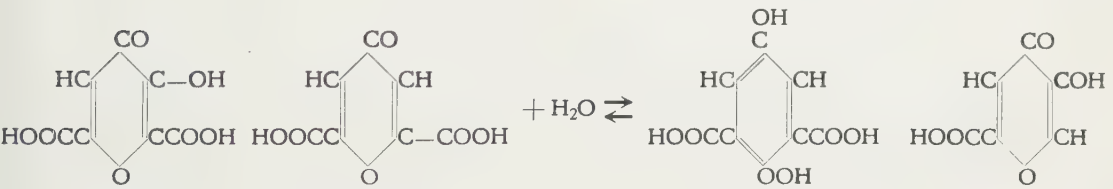
And, indeed, we see that the positive action begins with concentrations which are higher according as the hydroxy acid itself is the more split into ions. (Table VI.)

TABLE VI.

Hydroxy acid	Equivalent conductivity of 0.0005 m. solution	Conc. in gr. mol. $\times 10^{-4}$ whereby $\triangle$ becomes +
hydroxy 2 naphtoic acid 3	288	negative
„ 2 „ „ 1	228	2.1
„ 1 „ „ 2	248	3.2
Oxalic acid	195	
Sulphuric acid	368	

To the question which we propounded, as to how the negative behaviour of these naphtol carbonic acids, and especially of the 2.3 acid, in spite of the forming of borocomplexes, is to be explained, the following answer may be given: *because these acids are very strongly ionised* and are but very slightly soluble, so that in 0.5 molar boracic acid, almost only the monodiol boracic acids are present, which by loss of water will be partly converted into a non-electrolyte, just as boracic acid itself.

§ 3. Moreover, this explains the phenomena observed by VERKADE [5]) in the case of some γ-pyrone-carbonic acids, viz. meconic acid, chelidonic acid and comenic acid.



VERKADE showed that the two first acids dissolve more readily in 0.5 m. boracic acid than in pure water, but the third does not. Meconic acid and chelidonic acid, therefore, may easily form boro-complexes; nevertheless no increase in the conductivity was observed. (Table VII.)

The two first acids are equally strongly ionised as sulphuric acid; it is, therefore, quite comprehensible that in the concentration region examined

[5]) VERKADE, Rec. trav. Chim. **43**, 879 (1924).

TABLE VII.

Solubility and conductivity of γ-pyrone carbonic acids.

in H_2O at 25°		in 0.5 H_3BO_3 at 25°		
acid	g.m. p. L.	g.m. $\times 10^{-4}$ p. L.	equiv. con. calc. from V = 2048	increase of cond. in H_3BO_3
Meconic acid	8380	10000	367	negative
Chelidonic acid . . .	14500	15100	361	"
Comenic acid. . . .	5100	5100	176	"
Sulphuric acid . . .			368	

(V = 32 and upwards) there can be no question of an increase in the conductivity by boracic acid.

This extreme case joins itself very well with the behaviour of the ortho-naphtol carbonic acids. It is possible here that, even if didiol acid might be formed, the conductivity would not exceed that of the two strong acids.

It may be noted that VERKADE had taken into account this last-mentioned eventuality, as well as the forming of a derivative of the 3-valent borium (l.c. p. 891).

Since, with the highest concentrations examined (V = 32) an appreciable decrease in the conductivity was still found, and the concentrations, at a temperature of 25°, could not be raised to more than 0.05 m and 0.082 m (V = 20 and 12.2), it is not probable that the positive region can be reached experimentally. (See below.)

It would be interesting, nevertheless, to prepare salts of the boro-complexes from these acids.

There is still a second category of acids which, with respect to boracic acid, exhibit features analogous to the naphtol carbonic acids 1.2 and 2.1, i.e. a depression in a diluted solution of 0.5 m H_3BO_3, and in concentrated solution a (very considerable) increase in the conductivity, viz. the α-keto-acids [6]). (Table VIII.)

The high values of the increases may be explained from a hydratation of the COOH and the CO group, $CH_3C(OH)_2C(OH)_3$, as I assumed this previously, which is most favourable to the forming of boro-complexes. To explain the conversion into a negative effect with moderate dilutions, I further assumed that the carbonyl group would set free water. This rather improbable hypothesis may now be relinquished, and for it substituted the explanation that we have given for all the other strong acids.

As these keto acids are, namely, as strongly ionised as oxalic acid, their place beside the naphtolortho-carbonic acids is not merely reasonable.

[6]) Rec. trav. Chim. **39**, 387 (1919) and **41**, 568 (1921).

TABLE VIII.

Change of the conductivity of the α-keto-acids.

V	$C_6H_5COCOOH$	$(C_6H_3)_3C_6H_2COCOOH$	$C_6H_4CO(COOH)_2$	$CH_3COCOOH$	$(CH_3)_3CCOCOOH$
1	—	—	—	+ 32770	+ 30810
4	—	—	—	+ 6120	+ 1940
8	—	—	+ 1940 (V = 5)	+ 2980	—
15	+ 840	—	—	—	— 1740
16	—	—	—	+ 870	
20	—	—	+ 120	—	
30	+ 140	+ 460	—	+ 250 (V=32)	
60	+ 30	+ 60	—	−20 (V = 64)	
80			— 70	—	
120	— 40	— 20		—	

but necessary. Some of these acids present the great advantage that they dissolve readily in water, so that high concentrations could be studied experimentally, whereby, in their case, the region of the didiolboracic acid complexes could be reached with certainty. Very considerable increases in the conductivity were observed. From which it follows that the explanation of the reduction, or absence respectively, of any increase in conductivity, in the case of an excess of boracic acid (under conditions whereby the forming of boro-complexes may be expected and has even been demonstrated), viz. the liberation of a molecule H_2O from the monodiol boracic acid, attended by the forming of a non-electrolyte (see equation IV, page 470) is correct.

March 1940.

Geology. — *Remarks on the Geology of Colombia and Venezuela.* II. *The Areal Distribution of the Cretaceous.* By L. RUTTEN.

(Communicated at the meeting of March 30, 1940.)

It is necessary to begin with an exposition of some well-known facts concerning the Cretaceous of Venezuela. After the folding and denudation of the palaeozoic sediments, new sedimentation begins in middle mesozoic time. In the Sierra de Merida continental, red rocks are deposited; their age has been fixed as middle triasic or younger by the finding of Lepidotus (18). This red formation has been named differently by different authors; we use the oldest name of "Lagunillas conglomerate", given by SIEVERS (33). The rocks are chiefly conglomerates, sandstones and shales.

In 1930 I collected some samples near the bridge on the Rio Torbes, E. of Palmira, SW. Venezuela. They are a red, calcareous sandstone, a conglomeratic sandstone and a conglomerate. In the sandstone, the most common clastic elements are quartz grains; muscovite and plagioclase are rare. The pebbles in the conglomerates are: 1. quartz, 2. quartzites, often with strongly cataclastic quartzes, 3. fine-grained phyllites, 4. phyllite-quartzites, 5. sericite-quartzites, 6. fine-grained siliceous rocks, 7. quartz sandstones with strongly limonitized cement. Both granites and gneisses are absent in the samples.

The following facts indicate that the Lagunillas conglomerates probably have been deposited in isolated basins, separated by ridges: 1. their thickness varies quickly (9—3500 m); 2. their pebbles may belong to very different rocks of the basement, but at other points they may also have a quite local origin, being f.i. all derived from carboniferous sandstone; 3. there are localities in the Sierra de Merida, where the red formation is lacking and the mesozoic transgression begins with the Lower Cretaceous (22).

This "red formation" is very thick at the east side of the Sierra de Perijá (11); it is lacking in NE. Venezuela (10). It is widespread in the East Cordillera of Columbia and in the vicinity of the Sierra de Santa Marta; these regions will be treated later-on.

In Venezuela the Lagunillas formation is covered at some localities, conformably or pseudoconformably, by the Lower Cretaceous {E. Side S. de Perijá (11); Colon (17); parts of the Sierra de Merida (22)}; at other localities there is a slight unconformity between the two formations {parts of the Sierra de Merida (22)}. The Lower Cretaceous marks the beginning of a general transgression in Venezuela; the lowermost layers, containing Weichselia mantelli, may still have continental facies, but almost all the deposits are marine, reaching in places as high as the Upper Senonian {Rio Querecual (10); E. side Sierra de Perijá (11)}. The development of the formation is clearly cyclic: it begins with sandstones, which are followed

by limestones; the upper part of the formation consists of shales and cherts, which pass gradually into a coal-sandstone formation of lower

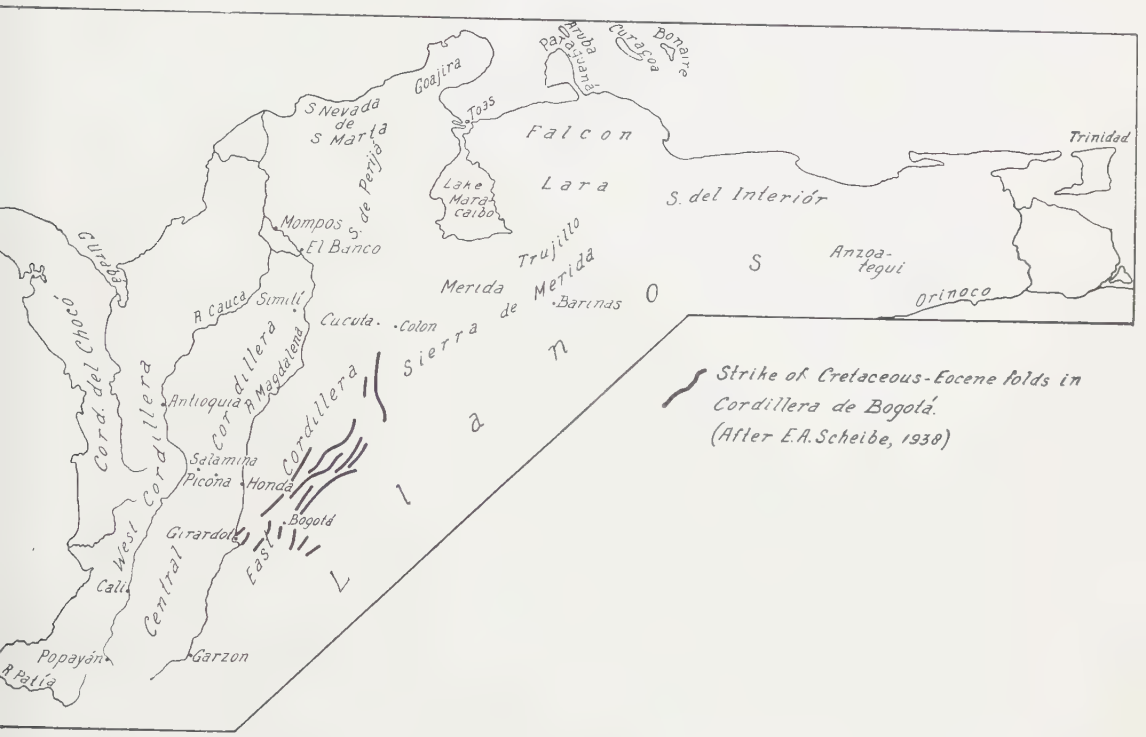


Fig. 1.

tertiary age. The thickness of the Cretaceous at different localities is as follows:

Rio Querecual, NE. Venezuela (10)	about	3300 m
Lara, between Carora and Valera (17)	more than	1400 m
Trujillo, N. of Mte. Carmelo (17)	about	1200 m
Region of Barinas (25)	about	450 m
Trujillo (Chejendé) (36)	about	1700 m
Merida (Zea) (17)	about	1200 m
Colon (17)	about	1750 m
East flank S. de Perijá (11)	about	4300 m

Without any doubt these thicknesses indicate that *the Cretaceous has a geosynclinal character*. The western continuation of the Venezuelan geosyncline is to be found in Colombia; the cretaceous sediments of Trinidad indicate its eastern continuation; their stratigraphy is, however, imperfectly known. At the southern side of the Sierra de Merida the thickness seems to diminish quickly (25): here the southern limit of the geosyncline seems to be not far away. This contention is corroborated by the discovery of "detritical Cretaceous" in a boring of Anzoategui, on the S. side of the Serranía del Interiór (6). There are equally data with

regard to the northern limit of the geosyncline. On Goajira peninsula the Cretaceous is strongly reduced and facially different from that in the geosyncline (35). On Toas island the Cogollo limestone transgresses on old granite; the lower-cretaceous sandstones are lacking here. On Paraguaná a boring passed from the basal conglomerate of the Miocene directly into metamorphic rocks (6). More to the N., on the islands of Aruba, Curaçoa and Bonaire, the Cretaceous has a facies, entirely different from that in the Venezuelan geosyncline. Therefore the N. limit of the geosyncline very probably lies near the N. coast of S. America. This limit, at least in W. Venezuela, was, however, not a limit between "geosyncline" and "great borderland", as has frequently been suggested (31). In the first place there has still been sedimentation in Goajira, on Toas and in Paraguaná, where L. KEHRER found cretaceous ammonites [1]). In the second place, marine Cretaceous is present on Aruba, Curaçoa and Bonaire, but in a totally different, porphyritic-diabasic facies. The thickness of these deposits is best known from Bonaire, where it exceeds 5000 m. The ridge between the Venezuelan and Bonairean geosyncline was no borderland, but only a zone of less sedimentation, or, at the best, a narrow geanticlinal area. Whether the northern foreland of the Venezuelan geosyncline in E. Venezuela may be regarded as a borderland, remains uncertain.

With regard to the Cretaceous of Venezuela, there are two points of great uncertainty. The first concerns the general form of the geosyncline. Did it comprise the whole of Venezuela between the Llanos and the N. coast or did it consist of two branches, a "Sierra de Perijá branch" and a "Sierra de Merida branch", and was the region, now covered by Lake Maracaibo and by the thick tertiary deposits of Falcón and N. Lara, during the Cretaceous a rising geanticlinal area? As far as I know, there is no drilling evidence which might elucidate this point. The second point of uncertainty concerns the cretaceous metamorphics; it has been treated by me in a former publication (26).

The "red formation" (Giron formation) of the East Cordillera of Colombia seems to comprise stratigraphically more than the "Lagunillas" of Venezuela. According to NOTESTEIN (in: 31) and L. KEHRER (18) — whose opinions are not shared by all other authors — the "Giron" consists of a lower division, comparable with "Lagunillas", which is covered slightly unconformably by an upper division, at the base of which Weichselia has been found, whilst in higher zones neocomian ammonites occur (5, 20). On the red to grey, lowermost cretaceous rocks follow marls, shales and limestones of the "Villeta" (12) which, in its turn, is covered by the "Guadeloupe", consisting chiefly of shales, siliceous shales and sandstones. The youngest marine fossils found until now in the "Guadeloupe"

[1]) The absence of Cretaceous in a boring in Paraguaná might even be secondary, due to postcretaceous denudation.

are of lower senonian age. In some parts of the East Cordillera, the "Guadeloupe" is covered slightly inconformably by the coal-bearing Eocene (29); in other places there seems to exist conformity between Guadeloupe and Eocene (27). The thickness of the Cretaceous in the East Cordillera has been estimated by various authors (1, 5, 19, 37) from 3000 to 7000 m! We may state that the *development of the Cretaceous in the East Cordillera is still more geosynclinal than in Venezuela.*

Our knowledge of the NE. part of Colombia — Sierra Nevada de Santa Marta and W. side of Sierra de Perijá — is shamefully imperfect and based entirely on a publication of SIEVERS from 1888 (32). The superposition at the W. side of the Sierra de Perijá is as follows: melaphyres, quartz-porphyries, volcanic tuffs and breccias, red conglomerates, red sandstones passing into yellow sandstones, cretaceous limestones (32, map, see: diagrammatic sections E. of Urumita and W. of Cerro Pintado). According to SIEVERS the thickness of the different elements is very great: conglomerates 800 m, sandstone 2000 m, limestones 800 m (32, p. 39, 40). The cretaceous limestones are not confined to the W. flank of the Sierra de Perijá; they pass the valley of the Rio Ranchería and are found in large extent at the E. side of the Sierra Nevada. SIEVERS even mentions the occurrence of presumably cretaceous limestones to the N. of the Sierra Nevada, and KARSTEN (16) has found belemnites in the Rio Palomino in the same region. The red sandstones also cross the valley between Sierra de Perijá and Sierra Nevada; they are found in a long and broad zone to the SE. of the Sierra Nevada between Alto de las Minas and Chantre. In this zone occur equally porphyrites and melaphyres, but here the age-relation between the effusive rocks and the sandstones is uncertain. KARSTEN mentions the occurrence of cretaceous deposits at the S. flank of the Sierra Nevada, but he does not give any details (16, p. 95).

Thus, we see that the *Cretaceous has still geosynclinal facies on the W. side of Sierra de Perijá and on the NE. side of the Sierra Nevada de Santa Marta*, and that the red formation seems to have a great extent at its S. side. Both — the Cretaceous and the "red formation" — have been strongly folded; the strike of the Cretaceous is NE., and the direction of folding to the NW.; the strike of the "red formation" at the S. side of the Sierra Nevada is W—E (31, p. 55).

Many authors are of the opinion that the Sierra Nevada is an old landmass (32, p. 47; 31, p. 646, 650). From the foregoing it must be concluded, that this opinion is hardly founded. From two or three sides the cretaceous limestones approach the Sierra Nevada, and there are no data to prove that the facies of the Cretaceous changes in the vicinity of the Sierra Nevada. There is not the slightest indication that the "red formation" and the Cretaceous did not extend originally over what is now the Sierra Nevada, and it may very well be that they have been eroded since then.

The Sierra Nevada is a mass of ancient rocks, but our geological knowledge of today does not favour the contention that it is an old land-mass.

The data concerning the Cretaceous of the Central Cordillera are scarce but interesting.

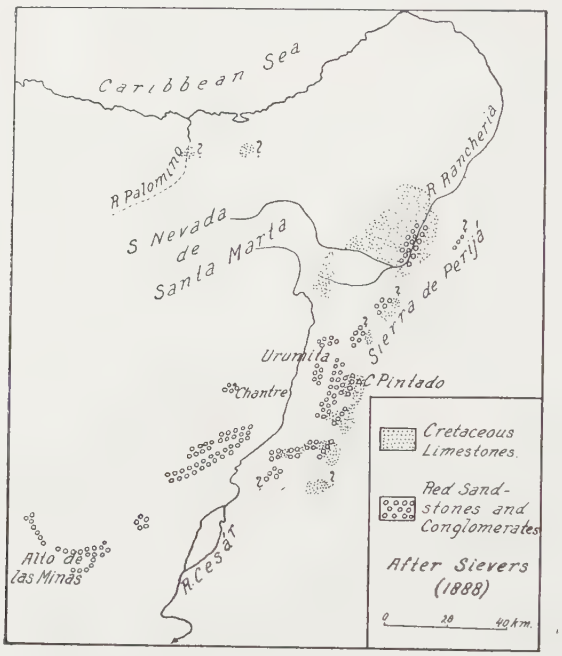


Fig. 2.

STUTZER (34) states, without giving details, that the formation is found "nur am Ostabfalle und im Norden" (p. 6).

WHEELER (37, p. 25) writes about the Cretaceous: "West of the Magdalena the only known outcrops are in the valleys near Simití, where the Lower Palmira rests directly upon the old crystalline rocks" [1]).

According to R. SCHEIBE (28) and R. LLERAS CODAZZI (24), cretaceous rocks are still to be found on the eastern flank of the Central Cordillera in the vicinity of Girardot.

All these data concern the E. flank of the Central Cordillera. I should like to stipulate that the different publications do not give any indications about a changing of the facies. On the contrary, WHEELER (l.c.) speaks of the "lower parts of the Palmira", which indicates that the facies is "normal". Nothing is known, however, in this area of "Giron", but it will be remembered that in the Venezuelan geosyncline the occurrence of the "red formation" is equally very irregular.

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[1]) WHEELER's text says "East of the Magdalena", but it certainly must be "West". "Palmira" comprises "Villeta" and "Guadeloupe".

A short publication of JOLEAUD (15) seems to indicate that normal cretaceous rocks occur rather far to the W. He writes:

"Sur le prolongement de la Cordillera Central, au Nord de la cluse de Mompos..... j'ai découvert une série de coteaux, jalonnant le raccord de la zone médiane de schistes cristallins de cette chaîne et de leur bordure mésozoique au massif de la Sierra Nevada de Santa Marta ..... Parmi les cerros qui marquent cette liaison tectonique les uns sont formés de diorites, les autres de calcaires crétacés; j'ai reconnu ces roches au Nord du Banco et aux environs de Chimichagua" (15, p. 1045).

There is an old publication, in which the occurrence of cretaceous sediments in the central part of the Central Cordillera has been described; nearly all the recent authors have ignored it. HETTNER–LINCK (13) mention the presence of "cretaceous" rocks from three zones: from Picona, near the highest part of the Central Cordillera; from the neighbourhood of Salamina, on the W. flank; and from Rio Sucio, in the West Cordillera. They write:

"Die Piconaschichten setzen sich aus weissem und braunem Sandstein, gelbem, rothem und grünlichgrauem Schieferthon, grauem Thonschiefer mit rother Verwitterungsfläche, schwarzem Kieselschiefer, einem gelben, kieseligen, dünnplattigen Gestein, welches dem von Monserrate bei Bogotá gleicht und vielleicht am besten als Wetzschiefer bezeichnet wird ..... zusammen. Versteinerungen habe ich nicht gefunden; ihre Aehnlichkeit mit den Gesteinen der Ostcordillere macht es jedoch wahrscheinlich, dass sie der Kreideformation angehören" (p. 210—211).

"Vom Cedral abwärts, ungefähr bei Salamina, finden sich hellgrauer Sandstein, Wetzschiefer, bunte Letter, weisser Thonschiefer, wie bei Pacho in der Ost-Cordillere, also sedimentäre Gesteine, vermuthlich cretaceischen Alters. Versteinerungen sind darin noch nicht nachgewiesen" (p. 212).

"Und hierauf folgen, etwas westlich vom Rio Sucio, Sandsteine, Schiefer u.s.w. der Kreideformation, welche nach NO bis NNO streichen und westlichen Einfall zeigen. Bei Salado ist einem System weissen Sandsteins ein 1¼ m mächtiges Kohlenflötz eingelagert, welches ganz an die Kohlenflötze von Bogotá erinnert" (p. 213).

The rocks from Rio Sucio, characterized by sandstones and coal, do certainly belong to the Tertiary of Antioquia. The data concerning the rocks of Salamina are not sufficient for an age-determination; they seem, however, to be very well comparable with the rocks of Picona. The rocks of the zone of Picona are described as being the same as those of Monserrate near Bogotá. According to HETTNER's map (12) Monserrate is entirely built-up by rocks of the "Guadeloupe", so that it is very probable that the rocks of the Picona-zone are Cretaceous. They are found in a zone of 12 km broad; the authors consider the zone to be a large anticline; the thickness of the formation must therefore be very great.

The data, given by HETTNER–LINCK, are so convincing, that I cannot see how we might avoid the conclusion that in the neighbourhood of Picona the Cretaceous is found in a facies, which is still comparable with that of the East Cordillera; the same is probable for the deposits near Salamina. The only other formation, with which these deposits might be compared, is the coal-bearing Tertiary of Antioquia. As this, however, is always

characterized by the presence of coal or conglomerates, which have not been mentioned from Picona and Salamina, I presume that a tertiary age of these rocks of the Central Cordillera is very improbable.

In the southernmost part of Colombia W. GROSSE (8) and W. KEHRER (21) have discovered cretaceous deposits with fossils, which may be regarded partly as belonging to the Central Cordillera, partly to the East Cordillera. It seems that here even "Giron" occurs; it is certain, that strata comparable with "Villeta" have been found. It is important that the facies of the Upper Cretaceous has changed, in so far as it is developed here in a "Porphyritical facies", which comprises effusive rocks, tuffs and cherts. According to GROSSE, the sequence is: 1. white sandstones, 2. Villeta, 3. Porphyrites etc.; the thickness is indicated as 3000 to 4000 m.

At Loma Hermosa, on the W. flank of the Central Cordillera (SE. of Antioquia), R. SCHEIBE (30) has found fossiliferous marls and sandstones with neocomian fossils, associated with porphyrites and their tuffs; it seems therefore to be certain that towards the W. the facies of the entire Cretaceous becomes volcanic. The Cretaceous of Loma Hermosa has a considerable thickness (7, 30). It is possible that on both sides of the Cauca river part of the Cretaceous is metamorphic (7, 21, 26).

The data for the remaining part of Colombia are meagre. H. KARSTEN (16, p. 85) regards part of the rocks of the West Cordillera as Cretaceous; he mentions the presence of cherts, sandstones and foraminiferal limestones in considerable thickness. Equally, K. ERMISCH (3, 4) mentions from the W. side of the West Cordillera "Plaener mit seinem Granitkontakthof"; from the same region and from the road from Bolívar to Carmen del Atrato he mentions the presence of cherts, which he considers as "older mesozoic". The data of KARSTEN and ERMISCH are too uncertain to be used in this discussion. HUBACH (14) mentions from the region of the Gulf of Urabá quartzites and shaly slates with ammonites and Inoceramus, which he considers to be of cretaceous age. I cannot see, how to avoid the conclusion that cretaceous rocks occur in this region. GROSSE (9) has found in the Patía region two non-fossiliferous formations, both consisting of porphyrites, cherts, tuffs etc. He regards the oldest — "Estratos de Chita" — as palaeozoic, the youngest as cretaceous. As far as I know, however, no contacts between the formations have been found, so that, in my opinion, they may very well have the same age. W. KEHRER also considers the "Estratos de Chita" as cretaceous; according to him a part of the Cretaceous of S. Colombia is perhaps metamorphic (21).

Our conclusions with regard to the occurrence of cretaceous deposits in the Central and West Cordillera are as follows: Cretaceous deposits with "eastcordilleran facies" are known from Simití and from the region W. of Girardot; possibly also from the region between El Banco and Mompos, from Picona and Salamina. In the neighbourhood of Garzon the lower parts of the Cretaceous still have "eastcordilleran facies", but the

facies of the upper part is here "porphyritic". On the W. flank of the Central Cordillera the Lower Cretaceous is developed in porphyritical facies. It is not improbable that cretaceous deposits in porphyritical facies are widespread in the West Cordillera and even in the Cordillera del Chocó. It is possible that cretaceous metamorphics occur in Antioquia and in S. Colombia. The Cretaceous of S. Colombia and Antioquia and the presumed Cretaceous of Picona and Salamina have considerable thickness. Although large parts of the Central Cordillera have lost their cover of cretaceous sediments, *it must be regarded as very probable that the cretaceous geosyncline comprised also the Central Cordillera and possibly the West Cordillera*. This conclusion is corroborated by the fact that the NE—SW direction of the cretaceous-eocene folds seems to cross the Magdalena river in the neighbourhood of Girardot, so that the Central Cordillera between Girardot and Popayán — of which very little is known — may very well contain relicts of such folds.

These conclusions are in sharp opposition with views generally held about the history of the Central (and West) Cordillera. We quote some of the modern authors:

"Nach meiner Ansicht war der grösste Teil der Mittelkordillere und Westkordillere zur Kreidezeit Festland" (STUTZER, 34, p. 6).

"The vast quantities of cretaceous clastics on the other hand, apparently came ..... in greatest amount from the Central Cordillera, which was rising during Cretaceous time" (SCHUCHERT, 31, p. 637). "It is probable that either in late Jurassic or early in Cretaceous time the Western and Central Cordilleras began to rise as high ridges with synclinal troughs to the east of each" (ibid. p. 670).

"In dieser Geosynklinale wurden Kreide und Eocän konkordant abgelagert, während sie in der Kolumbianischen Zentralkordillere, die m.E. das Resultat einer etwas älteren Faltung ist, nur rudimentär auftreten" (SCHÜRMANN, 31*a*, p. 377).

"An altogether different development took place in the Central Cordillera of Columbia, in the Western Cordillera and in the Sierra Nevada de Santa Marta. All the authors agree that Cretaceous deposits are almost completely lacking in these areas and that at numerous points Tertiary deposits lie unconformably over the basement rocks" (DE CIZANCOURT, 2, p. 212—213).

"During cretaceous time the Central Cordillera was a land mass representing the strand line of the sea that stretched over much of Eastern Colombia and Venezuela" (WHEELER, 37, p. 21).

The only one of these five authors, who gives arguments for his opinion, is DE CIZANCOURT; the other authors seem to base their opinion only on the fact that cretaceous deposits are lacking in the Central and West Cordillera. I should like to make the following remarks:

1. It has not been shown, that cretaceous deposits are lacking in Central and West Cordillera; great areas are unknown, and at several points the Cretaceous undoubtedly occurs.

2. Even, if the Cretaceous were lacking absolutely in the Central Cor-

dillera, this fact would not prove that this area has been "rising" in cretaceous time. It might be very well possible that the Central Cordillera had been uplifted more strongly than the East Cordillera in postcretaceous time, and that the cover of Cretaceous had been removed in old tertiary time (in that case the transgression of Tertiary over the basement would be comprehensable) or even in more recent time.

3. The following phenomena might be able to *prove* the "rising" of the Central Cordillera in cretaceous time:

a. the thinning-out of the Cretaceous from E. to W. on approaching the Central Cordillera,

b. the occurrence of coarser clastic material in the Cretaceous on approaching the margin of the Central Cordillera.

As far as I know, nothing of this kind has been discovered until now.

It may be that parts of the Central Cordillera which certainly have risen in *tertiary* time, represented a zone of reduced sedimentation during *cretaceous* time and that sedimentation, in a changed facies, increased again more to the W. (compare: Venezuelan geosyncline, N. Venezuela, Bonairean geosyncline), but this is only a possibility, for which, at present, no proof is available.

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Geology. — *Remarks on the Geology of Colombia and Venezuela.* III. *The Tertiary and the Caenozoic Tectonics.* By L. RUTTEN.

(Communicated at the meeting of March 30, 1940.)

Thanks to the very fine publications of the last five years we are well informed on the Tertiary of Venezuela, although, in detail, there is still much uncertainty. In the area of the Venezuelan cretaceous geosyncline there is conformity or — locally — pseudoconformity between Cretaceous and Tertiary. In contradistinction to the Cretaceous, the Eocene is for a great part continental, often characterized by the presence of coal. The Lower Eocene is sandy, the Upper Eocene is shaly. The thickness of the Eocene is as follows:

Táchira (20*a*)	less than	1900 m.
Barinas (24)		700 m.
Menegrande (40)	more than	2500 m.
N. foreland of Andes in Lara (34)		3300 m.
E. flank of S. de Perijá (15)	more than	2000 m.
Rio Querecual (NE. Venezuela) (14)		1800 m.

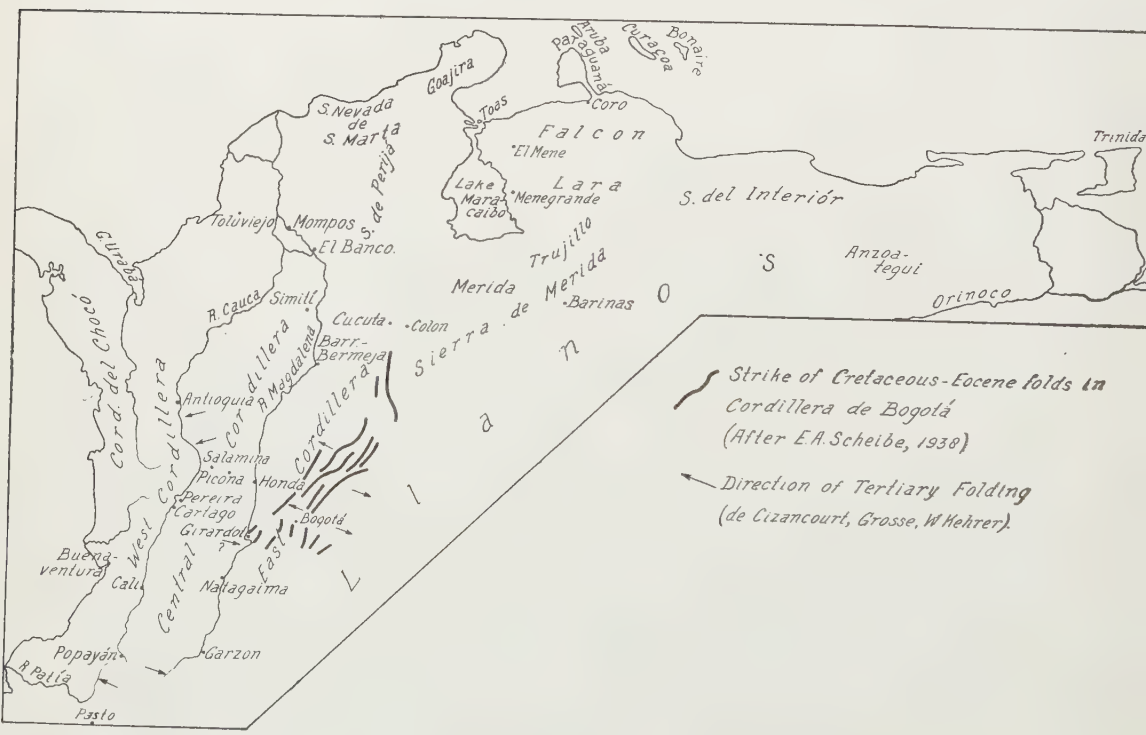

Fig. 3.

Where present in the S. de Merida, the thickness of the Eocene is still considerable. *Thus, the cretaceous geosyncline persists during the Eocene.* Near Barinas the southern boundary of the geosyncline was apparently not far from the S. foot of the Sierra de Merida: diminishing thickness. In general the formation is thicker at the N. border of the S. de Merida than in the mountains. The N. boundary of the geosyncline was not far from Toas, where the Eocene is reduced (23). In Lara the Eocene is somewhat metamorphic (33). *All in all there is great parallelism between the development of the Cretaceous and the Eocene in Venezuela.*

In the N. foreland of the Andes occurs an important intra-eocene angular unconformity (34) in Central Venezuela; the transgressing strata belong to the uppermost Eocene. An unconformity of about the same age is known from other localities. From upper-eocene time onward the Andes become a rising area, whereas large basins are developing to the N. and S. and in the depression between S. de Merida and East Cordillera of Colombia. Minor disconformities of locally changing age occur in these basins, but in each basin the period of strongest folding was the uppermost Tertiary. The thickness of the post-eocene deposits is indicated by the following figures:

Sierra de Merida	not present.
Táchira-Rubio (region between Andes a. East Cord.) (20a)	more than 2000 m.
Barinas (24)	4500 m.
E. flank Sierra de Perijá (15)	3000—5000 m.
Menegrande (only lower Post-Eocene) (40)	1000 m.
Oilfields E. of Lake Maracaibo (11)	less than 2500 m.
El Mene (12)	3000—5000 m.
Falcon and Coro (34)	less than 9000 m.
Rio Querecual (NE. Venezuela) (14)	5700 m.

It is clear that *these deposits are markedly geosynclinal.*

The conglomerates in the post-Eocene of Menegrande (40) and in the Eo-Oligocene of the boundary-area between Falcon and Lara (34) contain exclusively boulders of lower-eocene quartzites and sandstones; likewise, a very young conglomerate of El Mene, to the E. of Maracaibo, does not contain components of intrusive or metamorphic origin (25). Apparently the S. de Merida has not been stripped of its sedimentary cover until in subrecent times; *this indicates that its chief rising movement has occurred very late in geologic time.*

There are many indications that the northern post-eocene geosyncline ended not far from the N. coast: 1. the ?lower-pliocene Onia Beds, to the W. of Lake Maracaibo, are thinning-out from S. to N. (15); 2. different upper-tertiary deposits E. of Lake Maracaibo are equally thinning-out from S. to N.; 3. N. of El Mene the Eocene has been found in shallow borings (7); 4. in Paraguaná a boring passed directly from the Miocene

into the Eocene (7). The southern boundary of the Southern geosyncline is to be found N. of the Orinoco, as the Pre-cambrian is outcropping immediately S. of this river.

Thus, the cretaceous-eocene Venezuelan geosyncline passes, in post-eocene time, into two smaller geosynclines, where sediments of enormous thickness have been deposited. The area between these geosynclines, the S. de Merida, does not seem to have risen strongly before the Quaternary. The geology of the Venezuelan Tertiary is ruled by geosynclinal sedimentation and by folding; moreover, there occur certainly faults, and these become important in quaternary times: the coastline crosses obliquely the pliocene structures.

Conformity between the Cretaceous and the Eocene seems to occur also at some localities in the East Cordillera of Colombia (27, 39). For other localities a small *angular* unconformity has been indicated by R. SCHEIBE; the existence of an unconformity is moreover rendered probable by the following phenomena: 1. the absence of the uppermost cretaceous sandstones in some regions, 2. the occurrence of pebbles of upper cretaceous rocks in conglomerates of the Eocene (29, 31, 18). The thickness of the coal-bearing Eocene ("Guaduas") varies from 500 to 2000 m (27, 21, 1). The Guaduas formation occurs chiefly in the East Cordillera, but is also known from the Llanos (16) and from the Magdalena valley (4).

The considerable but changing thickness of the Guaduas, its presence in the East Cordillera and the bordering lowlands and the local occurrence of a pre-eocene unconformity do prove that, in general, the area of the East Cordillera was in eocene time a geosyncline, but that the geosynclinal movements were very different in different parts of the area and that even temporarily, local rising movements took place.

Even younger deposits are known in the East Cordillera: R. SCHEIBE (29) described a series of 300 m of sediments ("Barzalosa"), separated by an unconformity from the Guaduas, and W. KEHRER (21) mentions the presence of 700 m of "Bogotá-beds" in the neighbourhood of the capital.

Even in posteocene time sediments of considerable thickness have been deposited in what now is the East Cordillera.

Folded coal-bearing tertiary sediments ("Gualanday") have also been found on the E. flank of the Central Cordillera, in the neighbourhood of Girardot (28). In their upper parts they contain conglomerates with pebbles of quartzite, chert and -rarely- of upper cretaceous sediments. The Gualanday has been considered by R. SCHEIBE (28, 30) as post-eocene and as probable equivalent of "Barzalosa", chiefly because the quality of the Gualanday coal is worse than that of the Guaduas coal.

The Tertiary of the Middle Magdalena basin has been studied by many petroleum-geologists. According to WHEELER (42), whose publication is the most up to date one and based on the results of many colleagues, the

transgression begins near Simití with the Miocene. In the area of the Barranca Bermeja the thickness of the deposits, as well as their stratigraphical extent, increase rapidly from W. to E.; according to WHEELER, the maximal thickness of the Tertiary is 7500 m. The sum of the thicknesses, given by him for different parts of the formation, is considerably greater; about 12000 m!! Of the whole Tertiary only a part of the Eocene is marine. The lower Eocene consists chiefly of clayshale and coal; the whole tertiary sequence consists of sands, sandstones, clays and shales with some coal and conglomerates. The Miocene contains much magnetite and hornblende, indicating andesitic eruptions in the Hinterland. The thickness of the Eocene is 2300—?4300 m. There are two minor eocene unconformities and an important post-miocene unconformity. *The Tertiary of this part of the Magdalena basin has markedly geosynclinal character.*

The Tertiary of the neighbourhood of Garzon, which is partly tuffogeneous, is still very imperfectly known (22). In the upper Patía-basin GROSSE (10) found 430 m lower, coal-bearing Tertiary and 4000—7000 m(!!) upper Tertiary with intercalated lumachelles. Slightly folded sandstones with plant-remains are known from the region S. of Buenaventura (2).

The Tertiary of Antioquia is well known by the fine studies of GROSSE (9). The lower Tertiary, with a thickness of 1500—3400 m, consists of conglomerates, sandstones, shales and coal; the pebbles of the conglomerates are cherts, porphyrites etc.; the rare fossils are non-marine. The upper Tertiary, part of which is volcanic, has a thickness of at least 1000 m. There is a small unconformity between lower and upper Tertiary; the chief folding took place after the deposition of the upper Tertiary. In the neighbourhood of Cali, on the upper Cauca, tertiary coal-bearing strata are associated with Miogypsina-limestones (38)!!

In the Caribbean part of Colombia the Tertiary has chiefly or exclusively marine facies. The available data are badly localized, but they are sufficient for the conclusion that the thickness of the formation is at least 3000—4000 m and that the chief folding took place in late tertiary time (6, 1, 41). *The presence of geosynclinal tertiary deposits in this part of the republic cannot be doubted.*

We shall now try to get an idea of the tertiary palaeogeography of Colombia. We begin with some quotations of recent publications.

"That their principal ranges and intervening valleys had their beginning in pre-Tertiary time seems certain from the fact that the non-marine Tertiary deposits occupy intermontane basins in both Colombia and Venezuela. In Colombia these deposits extend inland along the valleys and among the mountains in a manner that suggests long and narrow inlets from the sea, leading first into estuarine and then into entirely non-marine conditions farther in the interior of the country" (F. M. ANDERSON, 1, p. 593).

"Die ersten Vorboten (des Einbruchs des Magdalenatales) erkennen wir bereits im Eocän. Nach dem Eocän, vermutlich zu Beginn des Oligocäns begann die eigentliche

Herausbildung des Grabens. Es erfolgte nicht nur ein Absinken, sondern auch ein Aufsteigen der beiderseitigen Randgebiete. Gleichzeitig ging auf der bis dahin vereinten Mittel- und Westkordillere ein zweiter, langgestreckter Grabeneinbruch nieder. Es ist dies der Cauca-Patiagraben, der von Cartago bis ..... Pasto reicht" (STUTZER, 39, p. 9—10).

"As we have said, after Laramid times breaking-down of the whole area on a large scale took place" (DE BÖCKH, in 8, p. 165).

"The rocks that border the Tertiary embayment of the middle and upper Magdalena were the source from which the Tertiary sediments were derived. These borderlands on the west side of the basin were the Central cordillera and on the east side, after the very early Tertiary at least the high cordillera of the Eastern Andes. These eastern and western forelands are thought to have been long, relatively narrow peninsulas extending out from the mainland which probably extended south and east from the present divide of the upper Magdalena and Amazon ..... The Tertiary deposits in the middle and upper Magdalena basin represent the detritus derived from the drainage systems of rivers flowing from the central and eastern Andes to the intervening Tertiary basin" (WHEELER, 42, p. 25—26).

SCHUCHERT (32, p. 662) cites, apparently with approval, the above given quotation of ANDERSON, but quotes afterwards the unpublished opinion of NOTESTEIN, that "ANDERSON's argument is weakened by the fact that Eocene sediments are thick and widely distributed on the plateau as well as in the intermontane valleys".

It is clear that ANDERSON and STUTZER, in their description of tertiary events, not only thought of the Magdalena basin, but that WHEELER's suggestions are restricted to this basin.

I have the impression that the different authors have been imposed unduly by the present topography and that they have projected backward this topography too far into former times. As to the Magdalena basin the following remarks may be made.

1. As argued above, it is not allowed to make a contrast between Magdalena basin and East Cordillera for eocene time. Both were chiefly sinking areas; in general the Magdalena basin was sinking more rapidly (thicker sediments). There were areas in the East Cordillera with temporarily rising movements (31), but there were equally areas in the basin, which, at least, temporarily did not sink (absence of lower Eocene in West part of the basin: WHEELER).

2. The East Cordillera was, in post-eocene times, not a regularly rising block: we have seen that locally thick younger sediments (21) are still present, and we do not know how much has been removed by erosion.

3. The bulk of the tertiary deposits in the Magdalena basin is clayey and sandy: a strongly rising East Cordillera would have furnished chiefly conglomerates. Moreover, part of the miocene sediments, rich in andesitic magnetite and hornblende, cannot have been derived from the East Cordillera (42). WHEELER indicates that important conglomerates occur in what he regards as upper Eocene and as Miocene and that in general the

coarseness of grain increases from W. to E.; this fact should prove the elevation of the East Cordillera. Four remarks may be made with regard to this contention. First, that WHEELER does not say that he has followed the increase of grain as far as the margin of the East Cordillera, where we should expect to find in every part of the Tertiary coarse conglomerates. Second, that — according to WHEELER — with the increase of coarseness of grain also the thickness of the formations increases. Now, it is certain that a zone of relatively thick sediments is also the zone of strongest sinking, and, if rivers coming from the S. (and not from the E.) brought the clastic material, they will have followed the zone of strongest sinking. Third that there are also strata in which the coarseness increases from E. to W. (42, p. 32). Fourth, that the conglomerates in the *Miocene* may indeed indicate rising of the East Cordillera, as we have seen that during younger tertiary times differential movements must have been much stronger than in the Eocene.

Thus, we see that there are no arguments which prove a contrast between basin and E. Cordillera during the Eocene, and that, for later times, there are indications that the strong contrast began not to develop before upper tertiary time.

At last, we should like to draw attention to the following quotation:

"It is thus apparent that there is an overlap of younger beds on older from east to west ..... and that there is a great thickening of similar beds in the section towards the east" (42, p. 26).

It is clear that the "overlaps" are much better in accordance with a broad synclinal or even geosynclinal area than with a "graben"-like zone, bounded by some major faults. We wish to point to the fact that JOLEAUD (19), without giving arguments, has regarded the broad valleys of Colombia not as "vallées d'effondrements", but as "vastes plaines néogènes, comme des aires synclinales".

It is more difficult to understand the conditions that governed the Central Cordillera during tertiary times. In contrast with WHEELER's (42, p. 39) opinion that the Central Cordillera has been elevated less than the East Cordillera we must state that, locally, already in eocene time important elevation must have taken place. WERENFELS (41) finds in the neighbourhood of Toluviejo eocene conglomerates with pebbles of diorite of 1 m diametre!! The facts, that near Simití the Miocene transgresses on the Cretaceous and at Puerto Berrio even on the basement, prove that in pre-miocene time this area was elevated above the sedimentation-level. The presence of pebbles in the "Gualanday" (see above, p. 486) proves that the Central Cordillera had a certain elevation during "Gualanday"-time. The existence of a marked unconformity between Cretaceous and Tertiary in Antioquia, the presence of important conglomerates at the base of the Tertiary and the fact that these conglomerates transgress on very different elements of the basement, do indicate that at the beginning of the tertiary

sedimentation *some parts* of the Central Cordillera must have been rather high. On the other hand the areas of sedimentation have certainly not been restricted to Magdalena- and Cauca-depression. Still now, the Gualanday extends for some distance into the Central Cordillera, and it is certain that the most elevated parts of these deposits have been removed by erosion during the ultimate rising of the Cordillera, so that originally they must have covered a greater extent. The Tertiary of Antioquia still now occurs up to 1900 m in the Central Cordillera, and it has been overthrusted from the E. by rocks of the basement. Before erosion, the Tertiary of this area must have extended rather far into the domain of the Central Cordillera. Higher-up the river Cauca tertiary rocks are chiefly known from the E. flank of the W. Cordillera; on the opposite W. flank of the Central Cordillera STUTZER (36) found chiefly young volcanic rocks; between Pereira and Cartago, however, E. dipping Tertiary. It is not improbable that the volcanics hide tertiary deposits, which would be in perfect agreement with what is known farther to the N., in Antioquia. At the E. flank of the Central Cordillera, in the area of Natagaima tertiary sediments may have wide distribution (35). If we go still farther southward, we find in the triangle between Garzon, Popayán and Pasto so many outcrops of folded Tertiary that it is very well possible that here the entire Central Cordillera has been an area of sedimentation during the Tertiary. The best illustration of our imperfect knowledge of the palaeogeography is the presence of Miogypsina-limestones near Cali (37). How did the sea reach this point of the republic? Certainly not via Antioquia and the Cauca depression, as there have been found only non-marine fossils in the Antioquia-Tertiary. It is in this connection of interest that GROSSE (10) found lumachelles in the Patía depression and that KARSTEN (20) has mentioned the presence of ?marine tertiary fossils even from the neighbourhood of Pasto. STUTZER's opinion that the Tertiary of the Cauca-Patía depression has been deposited in a "Graben" is certainly too simplistic. The rocks in what now is the Patía depression are strongly folded, in part even overthrusted, and they are not restricted to the present depression (10).

Our conclusion is that the Central Cordillera of to day has been during the Tertiary partly an important denudation-area, partly a very important sedimentation-area.

We have seen that Cauca- and Magdalena-depression have not been sedimentation-"Grabens". With their different tertiary unconformities, their overlapping and folded sediments they have much more the character of geosynclinal areas. If we regard them as such, *the relationship between the tertiary geology of Venezuela and Colombia becomes much greater.*

We now put the question whether the *actual* depressions of Cauca and Magdalena have the character of a "graben".

The contrast between the actual depression of Cauca-Patía and the

actual Cordillera must be very young. The arguments are the following. In both depressions there has been intense (post) upper-tertiary folding and overthrusting (9, 10). These movements are not restricted to the depressions, which must therefore be younger. The strike of the Cauca-valley S. of Antioquia is NE, the strike of the tertiary folds is NNW: the strike of the depression cuts the folds obliquely and must be younger. According to STUTZER (37) the limit of drainage between Cauca valley and the Pacific Ocean in the neighbourhood of Cali runs quite near to the valley. If the valley were an old depression there would certainly have developed important consequent rivers at its western flank. Now, the Cauca-Patía depression may be a very young "graben", but it might also be the result of strong selective erosion in a zone with soft sediments. It is, for the moment, impossible to choose between these alternatives, but there are two phenomena which may favour an erosive origin. S. from Antioquia the anticlinal axes, coming from the SSE, clearly have a dip to the NNW, i.e. towards the Cauca valley, and GROSSE mentions the existence of river terraces along the Cauca in heights up to 2000 m above the river (9)!!

We have quoted different authors who regard the Magdalena-basin as a relatively old "graben"; we might add STILLE (35), EUGSTER (5) and HARRISON (13). We have seen that there is strong probability that, during tertiary sedimentation, the basin was not a "graben" but a (geo)synclinal area. The actual Magdalena depression may be the result of faulting or of selective erosion; we have not found data which might prove one or the other origin.

The data on the character of the tertiary folding in Colombia are scarce but interesting. In the southern part of the Central Cordillera the movements are directed clearly W—E in the eastern part (22) and E—W in the western part (10). In both parts the existence of rather important overthrusts has been observed. In accordance herewith the direction of folding in the western part of the Central Cordillera in Antioquia is E—W. Some diagrammatic sections of R. SCHEIBE (28) suggest the existence of W—E direction of folding in the opposite part of the Cordillera, near Girardot. As for the East Cordillera, folding to the W. in its western part and to the E. in its eastern part has been defended by different authors (3, 32). If it is true, that Central and East Cordillera both show "symmetrical" folding this might indicate that the weaker zones have been underthrusted under the tectonically more elevated and therefore less mobile zones of the mountains. Data are, however, much too scarce for judging this problem. At all events, it would seem that the Sierra Nevada de Santa Marta has behaved differently (26).

Finally, we put again the question whether the Central- and West-Cordillera may be called — for tertiary time — "borderlands" with regard to the East Cordillera (see 26). The facts known of the Tertiary of the West Cordillera are so scarce that we must leave it out of the discussion.

As to the Central Cordillera I should say that it can — as a whole — not be regarded as a tertiary borderland and that it has been probably even been more mobile than the East Cordillera.

The arguments are as follows.

1. The presence of large, cretaceo-tertiary intrusives in different parts of the Central Cordillera (9, 10, 22, 35) is better in harmony with "mobility" than with a borderland.

2. Important upper-tertiary folding is known on a much greater scale from the Central Cordillera than from the East Cordillera (9, 22).

3. As well in Antioquia as in the southern part of the Central Cordillera the Tertiary has great extent and thickness.

Utrecht, March 1940.

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Palæontology. — *The fossil human remains discovered in Java by* Dr. G. H. R. VON KOENIGSWALD *and attributed by him to* Pithecanthropus erectus, *in reality remains of* Homo wadjakensis (syn. Homo soloensis). By Prof. EUG. DUBOIS.

(Communicated at the meeting of March 30, 1940.)

Concerning former discoveries of real human, pretended *Pithecanthropus* remains by VON KOENIGSWALD, published in the Proceedings: "Erste Mitteilung über einen fossilen Hominiden aus dem Altpleistocän Ostjavas". Vol. XXXIX, pp. 1000—1009. One Plate. (1936); "Ein Unterkieferfragment des Pithecanthropus". Vol. XL, pp. 883—893. One Plate. (1937), and "Ein neuer Pithecanthropus-Schädel". Vol. XLI, pp. 185—192. One Plate. (1938), I reported, shortly after the publications of those papers, in the Proceedings: "Racial Identity of *Homo soloensis* (including *Homo modjokertensis*) and *Sinanthropus pekinensis*". Vol. XXXIX, pp. 1180—1185. 4 illustrations. Three Plates. (1936); "The mandible recently described and attributed to the *Pithecanthropus* by G. H. R. VON KOENIGSWALD, compared with the mandible of *Pithecanthropus erectus* described in 1924 by EUG. DUBOIS". Vol. XLI, pp. 139—147. One illustration. Two Plates. (1938); "On the fossil human skull recently described and attributed to *Pithecanthropus erectus* by G. H. R. VON KOENIGSWALD". Vol. XLI, pp. 380—386. Two Plates. (1938).

The child skull of Modjokerto, the mandible and the skull of Sangiran, all of them undoubtedly human, I was convinced in my own mind to be morphologically distinct from *Pithecanthropus* and to have belonged at the same time to the proto-australian fossil man of Java, wellknown under the name of *Homo soloensis* (discovered in 1931 by Ir. W. F. F. OPPENOORTH), which is identical with the proto-australian *Homo wadjakensis* (which I fully described in the Proceedings, Vol. XXIII, pp. 1013—1051. Two Plates, 2 illustrations. (1920), and identified with *Homo soloensis* in "Nature", Vol. 130, p. 20. July 2, 1932, and also in "Man", Vol. XXXVII, January 1937, pp. 1—4).

VON KOENIGSWALD's next following discovery of a skull fragment at Sangiran (1938), "Pithecanthropus skull III", did not at all alter my view of the matter.

What, however, gives conclusive proof of VON KOENIGSWALD's misidentification is the discovery of his "Pithecanthropus skull IV", January 1939, again at Sangiran, "The latest and most important find of Pithecanthropus", according to WEIDENREICH ("Natural History". New York. January 1940, p. 34): the upper jaw, a large skull fragment, and part

of the lower jaw. Recently Dr. G. H. R. VON KOENIGSWALD, with Prof. FRANZ WEIDENREICH, briefly described and figured that upper jaw ("Nature", Vol. 144, p, 926. December 2, 1939), and, according to them, the upper jaw of *Homo wadjakensis* II displays a *most surprising resemblance* to this "Pithecanthropus" upper jaw.

Indeed! Also in every not restored part of this skull and mandible, in the ample and beautifully illustrated description by FRANZ WEIDENREICH ("Natural History". The Magazine of the American Museum of Natural History. January 1940, pp. 32—37), entitled "Man or Ape?", the resemblance to the proto-australian *Homo wadjakensis* II and *Homo soloensis* is striking. (Figs. 1, 2, 3, 4, and 5, after WEIDENREICH l.c.). Such it is in the characteristic slope of the nuchal plane of the occipital bone, in the contour of the not restored left half of the vault, etc. The latter may be compared with the right side-views of three Ngandong man skulls and the occipital view of one of them (*Homo soloensis* I, V, and VI. Fig. 6). Although the contours of those side-views differ much *inter se*, they, all of them, have not by far so flat and receding a frontal, and such a flattened parietal region above as this imitation-Pithecanthropus skull in the restored part. The restoration was, however, made under the supervision of Professor WEIDENREICH, on the basis of casts, and executed at the Palæontological Laboratory of the American Museum of Natural History, New York.

Nevertheless this "Pithecanthropus IV" skull once, at the time that it was complete, uncrushed and unweathered, *viz.* during life, has been, as it appears to me, an exact Wadjak-man, or Solo-man, skull.

Now, we have of *Homo wadjakensis* II not only the upper jaw, so surprisingly resembling that of VON KOENIGSWALD's "Pithecanthropus IV" skull, but also a nearly complete and well-preserved lower jaw. (Plate II. Right side-view; above. Front view; below). This, in its robustness, is comparable with the famous jaw of *Homo heidelbergensis*, although morphologically of a quite different, in a sense absolutely opposite, type. For the jaw of *Homo heidelbergensis* we may call the optimum form of the chinless *Homo neanderthalensis* jaw. The lower jaw of *Homo wadjakensis* II, on the contrary, possesses a most pronounced, we may call it an ideal, *Homo sapiens* chin, the jaw itself having the optimum form of that type.

Just that characteristic part of the lower jaw, however, is again lacking in VON KOENIGSWALD's last human fossil of Sangiran, as it was in his first, although larger, yet equally weathered and in this region no less incomplete lower jaw, so that WEIDENREICH (l c., p. 35) could say of "Pithecanthropus IV" skull: "No chin is present". But in both cases the bad preservation left room for (false) suppositions. In all probability a chin was originally present, not less pronounced than on the jaw of *Homo wadjakensis* II.

Needless to repeat that *Sinanthropus pekinensis* is another member of

the same proto-australian group, as sufficient arguments, in my opinion, are given in my papers: "The shape and the size of the brain in *Sinanthropus* and in *Pithecanthropus*". Proceedings, Vol. XXXVI, pp. 415—423. Four Plates. One illustration (1933), and "Racial identity of *Homo soloensis*.... and *Sinanthropus pekinensis*". l.c. (1936); further "On the fossil human skulls recently discovered in Java, and *Pithecanthropus erectus*". "Man", Vol. XXXVII, pp. 1—7. One Plate. 5 illustrations. January 1937.

(To be continued).

EUG. DUBOIS: *The fossil human remains discovered in Java by Dr.* G. H. R. VON KOENIGSWALD *and attributed by him to* Pithecanthropus erectus, *in reality remains of* Homo wadjakensis (syn. Homo soloensis).

PLATE I. "Pithecanthropus IV" and Homo soloensis I, V and VI

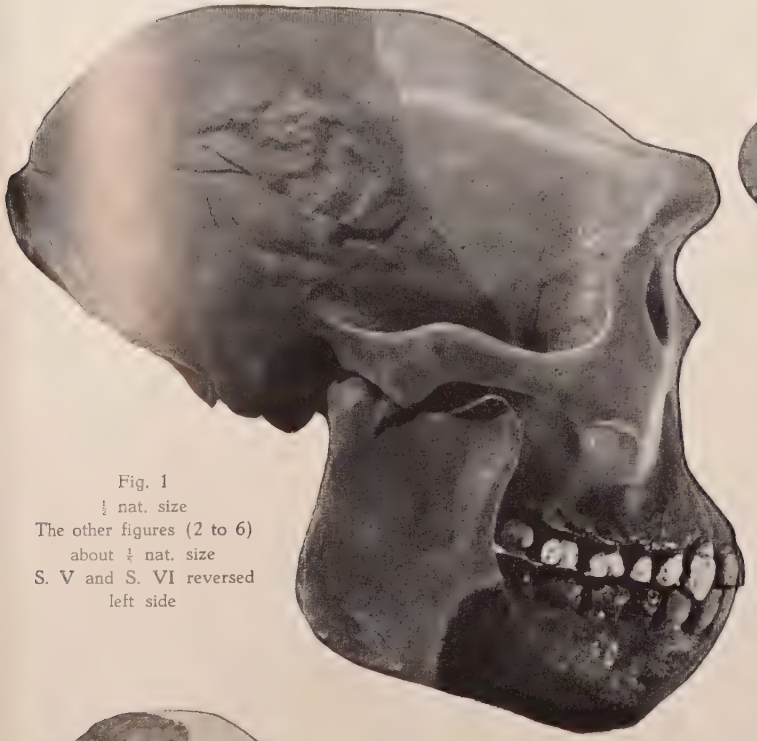


Fig. 1
½ nat. size
The other figures (2 to 6)
about ⅓ nat. size
S. V and S. VI reversed
left side

Fig. 2

Fig. 3

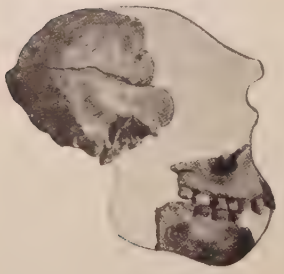
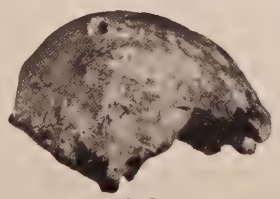
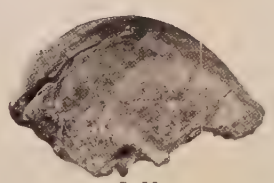


Fig. 4



S. I



S. V

Fig. 6

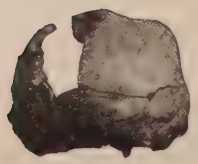
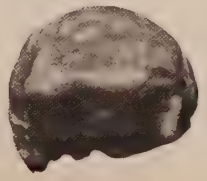
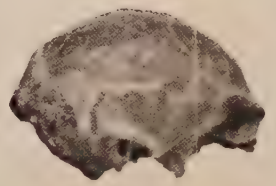


Fig. 5



S. I



S. VI

PLATE II. Homo wadjakensis II

Nat. size

Right-side view

Front view

Mathematics. — *Extension du problème de* DIRICHLET *pour ensembles quelconques.* Par A. F. MONNA. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of March 30, 1940.)

§ 1. Dans une communication précédente j'ai donné une généralisation du problème de DIRICHLET pour ensembles boréliens.[1]) Il s'agit maintenant d'une extension pour des ensembles bornés absolument quelconques. L'idée de cette extension est le résultat d'une correspondance avec M. BRELOT à Bordeaux. Nous ne considérons que le cas d'un espace euclidien à 3 dimensions; l'extension aux espaces supérieurs est évident.

§ 2. *Construction des solutions extérieures et intérieures.*

a. Soit d'abord $\Phi(P)$ une fonction continue sous-harmonique. Pour ensembles ouverts O on connait la solution généralisée de WIENER; on l'obtient en partant d'une suite d'ensembles réguliers ouverts croissants, tendant vers l'ensemble donné. Il est facile de voir que cette solution coïncide avec la fonction qu'on obtient par la voie suivante. Soit F un sous-ensemble fermé de O. Construisons la solution (d'après BRELOT) pour F et Φ. Alors on prend l'enveloppe supérieure de ces solutions pour tous les ensembles fermés contenus dans O.

On connait aussi la solution pour ensembles fermés F, obtenue en approximant F par une suite d'ensembles ouverts décroissants (utilisée déjà ci-dessus). On l'obtient aussi comme enveloppe inférieure des solutions de WIENER pour tous les ensembles ouverts contenant F.

Soit maintenant E un ensemble borné quelconque.

Définition. La solution intérieure $\underline{u}(P)$ *est l'enveloppe supérieure des solutions pour* Φ *et pour tous les ensembles fermés contenus dans* E. *La solution extérieure* $\bar{u}(P)$ *est l'enveloppe inférieure des solutions pour* Φ *et tous les ensembles ouverts contenant* E.

On montre que ces solutions sont harmoniques en tous les point intérieurs de E s'il y en a [2]).

Si $|\Phi| \leqq K$ on a $|\bar{u}|$ resp. $|\underline{u}| \leqq K$.

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[1]) Sur une généralisation du problème de DIRICHLET pour ensembles bornés mesurables B quelconques. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **42**, 745 (1939).

[2]) Voir: A. F. MONNA, Sur la capacité des ensembles. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **43**, 81 (1940).

b. Pour arriver à une définition pour des fonctions Φ continues quelconques, nous considérons d'abord le cas d'une fonction continue, différence de deux fonctions sous-harmoniques $\Phi = \Phi' - \Phi''$. Nous définions la solution intérieure et la solution extérieure comme différence des solutions correspondantes pour Φ' et Φ'':

$$\left.\begin{aligned} \bar{u} &= \bar{u}' - \bar{u}'' \\ \underline{u} &= \underline{u}' - \underline{u}'' \end{aligned}\right\}$$

On voit sans peine que la borne supérieure de $|\bar{u}|$ et de $|\underline{u}|$ est égale à la borne supérieure de $|\Phi|$ parce que cette propriété est vraie pour ensembles ouverts ou fermés.

Il s'ensuit que si la fonction continue Φ peut s'écrire de deux façons comme différence de fonctions sous-harmoniques, les solutions ainsi construites sont identiques. En effet, soient

$$\Phi = \Phi' - \Phi'' = \Phi_1' - \Phi_1''$$
$$\bar{u} = \bar{u}' - \bar{u}'' \ , \ \bar{u}_1 = \bar{u}_1' - \bar{u}_1''.$$

A $\Phi' + \Phi_1' - (\Phi'' + \Phi_1') \equiv 0$ correspond alors par définition $\bar{u}' + \bar{u}_1'' - (\bar{u}'' + \bar{u}_1') = \bar{u} - \bar{u}_1$ et cette fonction est donc identiquement nulle.

Ecrivons ensuite la fonction Φ continue quelconque comme limite d'une suite de polynomes, donc de différences de fonctions sous-harmoniques. Les solutions sont alors définies comme limite des suites des solutions correspondantes (l'existence de ces limites est immédiate). Il ne reste que de montrer que les solutions ainsi construites sont indépendantes des suites de polynomes. Ceci est facile. Soient

$$\Phi = \lim \Phi_n \ , \ \bar{u} = \lim \bar{u}_n$$
$$\Phi = \lim \Phi_n' \ , \ \bar{u}' = \lim \bar{u}_n'.$$

Alors $\bar{u}_n' - \bar{u}_n$, correspondant à $\Phi_n' - \Phi_n$, tend vers zéro quand $n \to \infty$; donc $\bar{u} = \bar{u}'$.

Remarques.

1. Les solutions sont fonctionelle linéaire de Φ.
2. Si $\Phi_1 \geqq \Phi_2$ on a $\bar{u}_1 \geqq \bar{u}_2$ et $\underline{u}_1 \geqq \underline{u}_2$.
3. Si Φ est sous-harmonique on a

$$\bar{u} \geqq \underline{u}.$$

4. La solution que j'ai introduite pour ensembles boréliens dans la communication mentionnée ci-dessus est identique à $\bar{u}$ si l'ensemble donné est limite d'une suite décroissante d'ensembles boréliens et identique à $\underline{u}$ si l'ensemble est limite d'une suite croissante.

5. Il est facile de généraliser la définition pour ensembles non bornés. On exige alors que les solutions sont nulle à l'infini (voir les notions de potentiel capacitaire extérieur et intérieur que j'ai défini récemment. Proc. Kon. Ned. Akad. v. Wetensch. Amsterdam 42.)

§ 3. *Stabilité et régularité.*

Un point-frontière Q de E appartenant à E sera appelé:

intérieurement stable si $\underline{u}(Q) = \Phi(Q)$,

extérieurement stable si $\bar{u}(Q) = \Phi(Q)$,

et cela quel que soit $\Phi(P)$ continu.

Un point-frontière Q de E n'appartenant pas à E sera appelé:

intérieurement régulier si $\lim_{P \to Q} \underline{u}(P) = \Phi(Q)$,

extérieurement régulier si $\lim_{P \to Q} \bar{u}(P) = \Phi(Q)$,

quel que soit $\Phi(P)$ continu.

Les critères de régularité et stabilité sont encore analogue à celles aux cas d'ensembles fermés ou ouverts.

1. Pour la régularité et stabilité il suffit de ne considérer que des fonctions Φ continues sous-harmoniques.

2. Pour la régularité ou stabilité (intérieurement ou extérieurement) il faut et il suffit que pour *une* fonction $\Phi(P)$, nulle en Q et > 0 ailleurs $\underline{u}(P)$ (respectivement $\bar{u}(P)$) $\to o$ avec $\overline{PQ}$ (ou, pour la stabilité, que $\underline{u}(Q)$ respectivement $\bar{u}(Q)$ est zéro).

3. Soit σ une sphére de centre Q et rayon ϱ suffisamment petit. Soit $(CE)_\varrho$ l'ensemble des points de CE situés sur ou dans σ. Désignons par $\bar{v}(P)$ et $\underline{v}(P)$ le potentiel capacitaire extérieur et le potentiel capacitaire intérieur de $(CE)_\varrho$. Pour que Q soit intérieurement régulier il faut et suffit alors que

$$\lim_{P \to Q} \bar{v}(P) = 1$$

Pour la régularité extérieure il faut et il suffit que

$$\lim_{P \to Q} \underline{v}(P) = 1.$$

Analogue pour la stabilité intérieure et extérieure [1].

Ce théorème est une conséquence du fait que de $\underline{w} \leqq 1 - \underline{v} \geqq \Phi$ sur la frontière de E (on prend $\underline{w} = 0$ aux points-frontière où $\underline{w}$ n'est pas défini) il suit que $\underline{w} \geqq \bar{u}$ où $\bar{u}$ signifie la solution extérieure pour Φ

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[1]) Voir aussi: A. F. MONNA, Sur les notions de „point stable" et „point régulier" dans le problème de DIRICHLET. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **43**, 87 (1940).

(analogue pour w et $\underline{u}$). Soit pour cela $O \supset E$. La portion A_ϱ de CO située sur ou dans σ est fermée et la solution v pour CA_ϱ et valeurs-frontière 1 est au plus égale à $\underline{v}$. On a donc $\phi \leqq w \leqq 1 - v$ sur la frontière de E. Donc, w étant harmonique dans O, $\bar{u}(P) \leqq w(P)$. En prenant la borne inférieure on obtient $\bar{u}(P) \leqq \underline{w}(P)$.

4. Soit Γ_n la sphère de centre Q et rayon λ^n où $0 < \lambda < 1$. Soit A_n la portion de CE comprise au sens large entre Γ_n et Γ_{n-1} et soient $\bar{c}_n$ et $\underline{c}_n$ les capacités extérieures et intérieures de A_n. Pour que Q soit intérieurement stable ou régulier il faut et suffit que la série

$$\sum_1^\infty \frac{\bar{c}_n}{\lambda^n}$$

diverge. Pour la stabilité ou régularité extérieure il faut et suffit que $\sum_1^\infty \frac{\underline{c}_n}{\lambda^n}$ diverge. On peut aussi exprimer ce critère par une intégrale. Soient $\bar{c}(\varrho)$ et $\underline{c}(\varrho)$ respectivement la capacité extérieure et la capacité intérieure de l'intersection de CE et de la sphère Γ de rayon ϱ. Il faut et il suffit alors que l'intégrale

$$\int_0^\varepsilon \frac{\bar{c}(\varrho)}{\varrho^2}\, d\varrho \quad \text{ou} \quad \int_0^\varepsilon \frac{\underline{c}(\varrho)}{\varrho^2}\, d\varrho$$

diverge.

5. Récemment M. BRELOT a introduit la notion nouvelle „d'ensemble effilé en un point" [1]). L'ensemble E est dit *effilé au point* P si $E \cap CP$ [2]) est vide ou à distance > 0 de P ou s'il existe au voisinage de P une fonction sous-harmonique u dont la valeur en P est supérieure à la plus grande limite en P de u sur $E \cap CP$. L'ensemble E est dit *effilé intérieurement en* P si tout sous-ensemble fermé de $E + P$ est effilé en P. Ces nouvelles notions sont étroitement liées aux notions de point irrégulier ou stable, extérieurement ou intérieurement. En effet, des critères nécessaires et suffisants que donne M. BRELOT il résulte immédiatement la relation suivante: *pour que le point-frontière Q de E soit irrégulier (ou instable si $Q \subset E$) intérieurement il faut et il suffit que CE soit effilé en Q; pour que Q soit extérieurement irrégulier (instable) il faut et il suffit que CE soit effilé intérieurement en Q.*

Conséquences.

1. Tout point Q extérieurement régulier (stable) est intérieurement régulier (stable) (ramener au cas de ϕ sous-harmonique).

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[1]) M. BRELOT, Sur la théorie moderne du potentiel. C. R. Ac. Sc. décembre 1939.

[2]) Intersection de E avec le complémentaire CP de P.

2. Soit E la fermeture de E et soit Φ sous-harmonique. Alors on a, si u^* désigne la solution (unique) pour $\bar{E}$ et Φ,

$$u^* \geqq \bar{u} \geqq u \geqq \Phi,$$

de sorte que si Q est stable pour $\bar{E}$, Q est régulier ou stable extérieurement (et donc aussi intérieurement). Il en résulte que les ensembles E^*_{esr} et E^*_{isr} des points-frontière stables ou réguliers extérieurement (intérieurement) sont partout dense sur E^*.

3. Les ensembles des points extérieurement ou intérieurement réguliers sont de capacité intérieure nulle. On voit cela en généralisant pour ensembles quelconques bornés la définition de M. DE LA VALLÉE POUSSIN de „point irrégulier sur un ensemble borélien", ce qui est d'ailleurs immédiate [1]). Tout point Q régulier en ce sens pour CE est régulier extérieurement (et donc aussi intérieurement) pour E et inversement. On montre alors, comme M. DE LA VALLÉE POUSSIN, que l'ensemble des points „réguliers" pour CE est de capacité intérieure nulle.

4. Soit Φ continu sous-harmonique. Alors les fonctions $(\bar{u}, \Phi)$ et $(\underline{u}, \Phi)$ sont à peu près sous-harmoniques. Pour les rendre sous-harmoniques il faut élever les valeurs de ces fonctions aux points irréguliers (resp. extér. et int.) aux valeurs de la plus grande limite de $\bar{u}$ resp. $\underline{u}$ sur E. Pour $(\bar{u}, \Phi)$ c'est immédiat, l'enveloppe inférieure d'une famille de fonctions sous-harmoniques étant sous-harmonique. Pour $(\underline{u}, \Phi)$ comparez les démonstrations de M. BRELOT.

§ 4. *Les extrémales sur E d'une fonction sous-harmonique.*

Le procédé pour ensembles boréliens conduit ici à deux extrémales d'une fonction sous-harmonique u: *l'extrémale extérieure* et *l'extrémale intérieure*. Nous les désignerons par $\overset{\circ}{\bar{u}}$ et $\overset{\circ}{\underline{u}}$. On a

$$\overset{\circ}{\bar{u}} \geqq \overset{\circ}{\underline{u}} \geqq u.$$

Aux points-frontière extérieurement stables $\overset{\circ}{\bar{u}} = u$ et aux points intérieurement stables $\overset{\circ}{\underline{u}} = u$. Les fonctions $(\overset{\circ}{\bar{u}}, u)$ et $(\overset{\circ}{\underline{u}}, u)$ sont à peu près sous-harmoniques et deviennent sous-harmoniques lorsqu'on élève leurs valeurs aux points irréguliers (extérieurement ou intérieurement) aux plus grandes limites de $\overset{\circ}{\bar{u}}$ ou $\overset{\circ}{\underline{u}}$ en ces points.

Propriétés.

1. $\overset{\circ}{\bar{u}}$ resp. $\overset{\circ}{\underline{u}}$ est l'enveloppe inférieure de la famille des majorantes que sont les solutions extérieures resp. intérieures pour distributions continues $\geqq u$ sur E^*.

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[1]) CH. DE LA VALLÉE POUSSIN, Points irréguliers, Détermination des masses par les potentiels. Bull. Acad. royale des Sciences de Belgique t. 24, fasc. 6, 7, 11 (1938).

2. $\overset{\circ}{\bar{u}}$ est l'enveloppe inférieure des extrémales (donc ici meilleures majorantes harmoniques) pour tous les ensembles ouverts contenant E. $\overset{\circ}{u}$ est au moins égal à l'enveloppe supérieure des extrémales pour tous les sous-ensembles fermés de E (comparez avec la plus petite majorante harmonique pour ensembles ouverts).

3. De toutes les fonctions définies sur E et qui, prolongées par u, sont sous-harmoniques sauf éventuellement aux points-frontière irréguliers extérieurement, $\overset{\circ}{\bar{u}}$ est la plus grande.

Soit pour cela O un ensemble ouvert contenant E et soit w une fonction de la famille; posons $W = \widehat{(w,u)}$. Aux points-frontière de O situés dans $\bar{E}$ on a $W = u$. Aux points-frontière de O qui sont également point-frontière de E mais n'étant pas extérieurement irréguliers pour E, on a $\overline{\lim}\, W(P) \leqq u$ $(P \subset O)$. Soit alors ϕ continu $\geqq u$ sur O^*. Il s'ensuit que W vaut au plus la solution pour ϕ et O. Donc aussi $W \leqq \overset{\circ}{u}_0$, où $\overset{\circ}{u}_0$ désigne l'extrémale de u pour O. En tenant compte de la propriété précédente, on obtient $W \leqq \overset{\circ}{\bar{u}}$ et puisque $W = w$ sur E, on a $w \leqq \overset{\circ}{\bar{u}}$. L'extrémale extérieure est donc bien la plus grande fonction de la famille.

Propriété analogue pour l'extrémale intérieure. Il faut seulement remplacer dans la définition le mot extérieurement par intérieurement (la démonstration est plus délicate). Remarquons que la famille F_i ainsi définie est contenue dans la famille F_e définie ci-dessus puisque tout point irrégulier intérieurement est aussi irrégulier extérieurement. Ceci est en accord avec l'inégalité $\overset{\circ}{\bar{u}} \geqq \overset{\circ}{\underline{u}}$.

§ 5. *Représentation intégrale des solutions.*

Puisque les solutions sont des fonctionnelles linéaires de ϕ continu, on peut écrire comme il est bien connu

$$\bar{u}(P) = \int_{E^*} \phi(Q)\, d\bar{\mu}^P(e_Q),$$

$$\underline{u}(P) = \int_{E^*} \phi(Q)\, d\underline{\mu}^P(e_Q),$$

où $\bar{\mu}^P$ et $\underline{\mu}^P$ sont des fonctions positives d'ensemble borélien sur la frontière E^* qui dépendent du point P de E. On a $\bar{\mu}^P(E^*) = \underline{\mu}^P(E^*) = 1$. Pour e fixe, $\bar{\mu}^P(e)$ est une fonction de P qu'on obtient ainsi. La fonction égale à 1 sur e et à 0 ailleurs sur E^* est limite d'une suite de fonctions continues f_n. Alors $\bar{\mu}^P(e)$ est la limite des solutions extérieures pour les f_n quand $n \to \infty$ (analogue pour $\underline{\mu}^P(e)$). On peut caractériser $\bar{\mu}^P$ et $\underline{\mu}^P$ aussi d'un autre façon. Remarquons pour cela d'abord qu'on peut dire que ces deux fonctions d'ensembles déterminent sur E^* deux distributions

de masses positives en total 1. Il s'agit maintenant de déterminer le potentiel de ces deux distributions.

1. Soit P un point de E et R_e un point de $C\bar{E}$ et prenons $\Phi(P)=\dfrac{1}{R_e P}$. Alors evidemment $\bar{u}(P)=\underline{u}(P)=\dfrac{1}{R_e P}$. Donc

$$\left.\begin{aligned}\int_{E^*}\frac{1}{R_e Q}\,d\bar{\mu}^P(e_Q)&=\frac{1}{R_e P}\\ \int_{E^*}\frac{1}{R_e Q}\,d\underline{\mu}^P(e_Q)&=\frac{1}{R_e P}\end{aligned}\right\}$$

Hors $\bar{E}$ le potentiel de $\bar{\mu}^P$ et celui de $\underline{\mu}^P$ pour P fixe est donc égal au potentiel d'une masse 1 en P.

2. Déterminons maintenant le potentiel sur $\bar{E}$. Il faut distinguer les deux cas.

a. Soit O un ensemble ouvert contenant E et soit $\mu_O^{P_0}(e)$ la distribution sur la frontière O^* de O, obtenue par balayage de la masse 1 en P_0 de E (P_0 fixe) Alors on sait que

$$-\int_{O^*}\frac{d\mu_O^{P_0}(e_Q)}{QR}=\overbrace{\left(\left(\frac{\overset{\circ}{-1}}{P_0R}\right)_O,\frac{-1}{P_0R}\right)},$$

où $\left(\dfrac{\overset{\circ}{-1}}{P_0R}\right)_O$ désigne l'extrémale (la meilleure majorante harmonique) de $-\dfrac{1}{P_0R}$ pour O. Afin de déterminer l'enveloppe inférieure de l'intégrale à gauche en un point R fixe pour tous les $O\supset E$, prenons la moyenne spatiale de la fontion à intégrer dans une sphère de centre Q et rayon $1/n$; on la désigne par

$$\int_{O^*}\left(\frac{-1}{QR}\right)_{1/n}d\mu_O^{P_0}(e_Q).$$

La fonction sous l'intégrale étant alors continue et sous-harmonique, l'enveloppe inférieure est par définition

$$\int_{E^*}\left(\frac{-1}{QR}\right)_{1/n}d\bar{\mu}^{P_0}(e_Q).$$

Il en résulte pour $n\to\infty$ que l'enveloppe cherchée est égale à

$$\int_{E^*}\frac{-1}{QR}\,d\bar{\mu}^{P_0}(e_Q).$$

En conséquence des propriétés de l'extrémale (§ 4) l'enveloppe inférieure de la fonction à droite est sur E égale à $\left(\overset{\circ}{\overline{\frac{-1}{P_0 R}}}\right)$, c. à. d. l'extrémale extérieure de la fonction sous-harmonique $\frac{-1}{P_0 R}$. Hors $\bar{E}$ l'enveloppe est $\frac{-1}{P_0 R}$, conforme au résultat du n° 1. Nous allons établir qu'aux points Q_0 réguliers extérieurement la valeur de l'enveloppe inférieure est encore $\frac{-1}{P_0 Q_0}$. Désignons, pour abréger, la fonction dont nous cherchons l'enveloppe par $U_O(R)$. Puisque $U_O(R) \geqq \frac{-1}{P_0 R}$ (propriétés des extrémales!) $U(Q)$ est au moins égal à $\frac{-1}{P_0 Q_0}$. Soit σ_1 une sphère (fermée) de centre Q_0 et de rayon $< \overline{P_0 Q_0}$. Soit σ_2 concentrique plus petite. Prenons alors une fonction continue $\Phi(R)$ telle que $\Phi(Q_0) = -\frac{1}{P_0 Q_0}$, $\Phi(R) > -\frac{1}{P_0 R}$ ailleurs et telle que Φ est supérieur hors σ_2 à la borne supérieure de U_O. Soit O' un ensemble ouvert contenant $E\sigma_2$ et contenu dans σ_1. O' peut avoir des points-frontière en commun avec E. Or, si un tel point est régulier pour O', on voit aisément que la borne inférieure de U_O en ce point est égale à $-\frac{1}{P_0 Q}$ (la borne est alors même atteinte pour un O spécial). On voit alors que $\Phi \leqq U_O$ au moins en tous les points-frontière reguliers de O', c. à. d. à peu près partout sur la frontière de O'. Il en résulte que la solution pour O' et Φ vaut au moins U_O. Faisons alors tendre O' vers $E\sigma_2$: on obtient alors la solution extérieure pour $E\sigma_2$ et Φ et puisque Q_0 est aussi extérieurement régulier pour $E\sigma_2$, cette fonction tend vers $-\frac{1}{P_0 Q_0}$ en Q_0. Il en résulte évidemment que la borne inférieure de U_O en Q_0 ne peut rester $> -\frac{1}{P_0 Q_0}$ et elle vaut donc $-\frac{1}{P_0 Q_0}$, ce qu'il fallait montrer.

b. Soit F un ensemble fermé contenu dans E et soit $\mu_F^{P_0}(e)$ la distribution sur la frontière F^* de F, obtenue par balayage de la masse 1 en P_0 de E (P_0 fixe). On a donc

$$-\int_{F^*} \frac{d\mu_F^{P_0}(e_Q)}{QR} = \left(\left(\overset{\circ}{\frac{-1}{P_0 R}}\right)_F, \frac{-1}{P_0 R}\right),$$

où $\left(\overset{\circ}{\frac{-1}{P_0 R}}\right)_F$ désigne l'extrémale de $\frac{-1}{P_0 R}$ pour F. On voit encore que l'enveloppe supérieure de cette intégrale vaut — sauf aux points d'un

ensemble de capacité nulle; on verra que ce sont les points intérieurement irréguliers —

$$-\int_{E^*}\frac{d\mu^{P_0}(e_Q)}{QR}.$$

Pour $R \subset E$, il résulte alors de l'égalité précédente en conséquence de la propriété n° 2 du § 4, que

$$\frac{-1}{P_0 R} \leqq -\int_{E^*}\frac{d\mu^{P_0}(e_Q)}{QR} \leqq \underline{\left(\frac{\overset{\circ}{-1}}{P_0 R}\right)}.$$

Supposons qu'en un point R_0 on ait à droite l'inégalité. Prenons alors la fonction $V_{1/n}(R)$, déduite de l'intégrale — que nous désignons par $V(R)$ — par médiation spatiale dans une sphère de rayon $1/n$. On peut prendre n assez grand pour que

$$\frac{-1}{P_0 R_0} \leqq -V_{1/n}(R_0) < \underline{\left(\frac{\overset{\circ}{-1}}{P_0 R_0}\right)}.$$

Or, $-V_{1/n}(R)$ est sous-harmonique, partout continue et est la solution intérieure de E pour ses propres valeurs sur E^* puisque cette fonction est l'enveloppe supérieure pour tous les F de la famille des solutions $\left(\int\frac{d\mu_F^{P_0}}{QR}\right)_{1/n}$. On aurait ainsi une contradiction avec la première propriété du § 4 et on a donc sur E

$$-\int_{E^*}\frac{d\mu^{P_0}(e_Q)}{QR} = \underline{\left(\frac{\overset{\circ}{-1}}{P_0 R}\right)}.$$

En ce cas il est immédiat que le potentiel vaut $\frac{-1}{P_0 R}$ hors E et aux points intérieurement réguliers.

Conséquences.

1. Considérons les fonctions à peu près sous-harmoniques $(\overset{\circ}{\bar{u}}, u)$ et $(\overset{\circ}{\underline{u}}, u)$ où $u = -\frac{1}{P_0 R}$. On ne peut les rendre sous-harmoniques que d'une seule manière: $(\overset{\circ}{\bar{u}}, u)$ et $(\overset{\circ}{\underline{u}}, u)$. Ces dernières fonctions sont donc, à une fonction harmonique près — dont on montre facilement qu'elle est nulle en notre cas — égales aux potentiels de deux distributions de masses négatives, uniquement déterminées par les potentiels. Il résulte des

démonstrations précédentes que ces distributions sont justement $-\overline{\mu}^{P_0}$ et $-\underline{\mu}^{P_0}$. On a donc

$$\left.\begin{aligned} -\int_{E^*} \frac{d\,\overline{\mu}^{P_0}(e_Q)}{QR} &= \left(\overset{\circ}{\left(\frac{-1}{P_0 R}\right)}, \frac{-1}{P_0 R}\right) \\ -\int_{E^*} \frac{d\,\underline{\mu}^{P_0}(e_Q)}{QR} &= \left(\overset{\circ}{\left(\frac{-1}{P_0 R}\right)}, \frac{-1}{P_0 R}\right) \end{aligned}\right\}$$

Le passage de la masse -1 en P_0 de E à la distribution $-\overline{\mu}^{P_0}(e)$ sur E^* ($-\underline{\mu}^{P_0}(e)$ sur E^*) conserve donc le potentiel hors $\overline{E}$ et aux points stables ou réguliers extérieurement (intérieurement) et le rend maximum sur E. On peut appeler ces deux opérations *„extrémisation extérieure"* et *„extrémisation intérieure"*.

2. De la formule générale pour les solutions on déduit les critères suivants: Pour que le point-frontière Q de E soit extérieurement ou intérieurement stable il faut et il suffit que $\overline{\mu}^P$ ou $\underline{\mu}^P$ se réduit à la masse 1 en Q. Pour que le point-frontière Q non de E soit extérieurement ou intérieurement régulier il faut et suffit que de $\overline{\mu}^P$ ou $\underline{\mu}^P$ tend vers la masse 1 en Q quand P de E tend vers Q.

3. Pour les deux extrémales d'une fonction sous-harmonique $u(P)$ on a les formules

$$\overset{\circ}{\overline{u}}(P) = \int_{E^*} u(Q)\, d\,\overline{\mu}^P(e_Q).$$

$$\overset{\circ}{\underline{u}}(P) = \int_{E^*} u(Q)\, d\,\underline{\mu}^P(e_Q).$$

En particulier si $u(P) = -\frac{1}{RP}$ (R fixe),

$$\overset{\circ}{\overline{u}}(P) = -\int_{E^*} \frac{1}{RQ}\, d\,\overline{\mu}^P(e_Q)$$

et analogue pour $\overset{\circ}{\underline{u}}(P)$. Puisque $\overset{\circ}{\overline{u}}(P)$ vaut aussi le potentiel en P de $\overline{\mu}^R$, on a donc

$$\int_{E^*} \frac{d\,\overline{\mu}^P(e_Q)}{RQ} = \int_{E^*} \frac{d\,\overline{\mu}^R(e_Q)}{PQ} \qquad (P \text{ et } R \text{ sur } E).$$

et analogue pour $\underline{\mu}^P$.

4. L'extension de la définition pour distributions quelconques de masses négatives va maintenant sans peine. Soit θ une telle distribution sur E [1]). Considérons alors la distribution sur E^*, définie par

$$\bar{\mu}(e) = \int_E \bar{\mu}^P(e)\, d\theta(P), \qquad (P \subset E)$$

où e désigne un sous ensemble borélien quelconque de E^*. On montre alors aisément en utilisant la propriété de symmétrie précédente (calcul sur des intégrales de Stieltjes), que le potentiel de $\bar{\mu}(e)$ vaut celui de θ hors $\overline{E}$ et aux points stables et reguliers extérieurement; sur E ce potentiel vaut l'extrémale extérieure $\overset{\circ}{\bar{\mu}}$ sur E du potentiel u de θ, donc y est rendu maximum. Le passage de θ à $\bar{\mu}$ sera appelé. *„l'extrémisation extérieure"*. Dèfinition analogue de *l'extrémisation intérieure.*

Il résulte aussi de cette démonstration *qu'on a* $\overset{\circ}{\bar{u}} = u$ *si* θ *ne charge que l'ensemble* E^*_{es} *des points extérieurement stables.* En effet, on verra sans peine que

$$\overset{\circ}{\bar{u}}(P) = \int_E \alpha^P(R)\, d\theta(R), \qquad (P \text{ et } R \text{ sur } E)$$

où $\bar{\alpha}^P(R)$ désigne le potentiel de $\bar{\mu}^P$ en R. Sur E^*_{es} on a $\bar{\alpha}^P(R) = \dfrac{1}{PR}$ et donc

$$\overset{\circ}{\bar{u}}(P) = \int_{E^*_{es}} \frac{d\theta(R)}{PR} = u(P).$$

Puisque de plus l'extrémisation conserve le potentiel aux points réguliers, il suit que le potentiel est conservé en ce cas à peu près partout. Il en résulte que *l'extrémisation extèrieure ne change pas des masses seulement portées par* E^*_{es} (*analogue pour l'extrémisation intérieure*).

On montre encore en partant des formules générales du n° 3 la propriéte suivante:

Soit ν *une distribution de masses négatives sur l'ensemble* E^*_{er} *des points extérieurement réguliers et soit* u *le potentiel de* ν. *Alors on a* $\overset{\circ}{\bar{u}} = u$.

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[1]) Pour la définition de „distribution sur un ensemble quelconque" voir: O. FROSTMAN, Potentiel d'équilibre et capacité des ensembles. Meddelanden från Lunds Universitets Mat. Sem. **3** (1935).

5. Nous définirons la fonction de Green extérieure et la fonction de Green intérieure comme il suit:

$$\overline{G}(P,R)=\frac{1}{PR}-\int_{E^*}\frac{d\bar{\mu}^P(e_Q)}{QR}$$

$$\underline{G}(P,R)=\frac{1}{PR}-\int_{E^*}\frac{d\underline{\mu}^P(e_Q)}{QR}.$$

Ces fonctions sont non-négatives et symmétriques. On a le critère suivant: Le point Q_0 de E^* est stable extérieurement (ou intérieurement) si $\overline{G}(Q_0,R)=0$ (ou $\underline{G}(Q_0,R)=0$) à peu près partout pour R dans E [1]) (pour la régularitè il faut prendre $\lim_{P\to Q_0}\overline{G}(P,R)=0$).

En effet, d'après ce qui précède,

$$\overline{G}(Q_0,R)=\frac{1}{Q_0R}-\int\frac{d\bar{\mu}^{Q_0}(e_Q)}{QR}$$

est zéro hors $\overline{E}$ et aux points réguliers et stables. Les points irréguliers formant un ensemble de capacité nulle, $\overline{G}(Q_0,R)$ est zéro à peu près partout et $\bar{\mu}^{Q_0}$ se réduit donc à la masse 1 en Q_0, c.à d. Q_0 est extérieurement stable (analogue pour la régularité comme on voit aisement).

Inversement, si le point Q est extérieurement ou intérieurement stable, $\overline{G}(Q,R)$ ou $\underline{G}(Q.R)$ est zéro quel que soit R dans E (de même pour la régularite).

On en conclut que les ensembles des points intérieurement ou extérieurement instables ou irréguliers sont des réunions dénombrables d'ensembles fermés. En effet, pour un tel point Q il existe certainement un point R de E tel que $\overline{G}(Q,R)>0$ (ou $\overline{\lim_{P\to Q}}\,\overline{G}(P,R)>0$ etc.) et inversement s'il existe un point R rendant vraie cette inégalité, le point-frontière est instable (etc.) (il se peut que le même point R ne suffit pas pour chaque Q mais cela n'importe). L'ensemble des points extérieurement instables est donc la somme des ensembles fermés où $\overline{G}(Q,R)\geqq\frac{1}{n}$ $(n=1;\ 2,\ldots)$.

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[1]) Remarquons qu'il existent certainement des points R dans E tels que $\overline{G}(Q_0,R)$ y est zéro, si l'ensemble des points stables n'est pas vide (et contient des points différent de Q_0). En effet, en un tel point stable R_0, l'extrémale de $\frac{-1}{Q_0R}$ (masse -1 en Q_0), c.àd. l'intégrale figurant dans la définition de $\overline{G}_{,,}$ vaut $\frac{-1}{Q_0R_0}$ et $\overline{G}(Q_0,R_0)=0$. Au cas général il ne suffit donc pas, comme il est vrai pour ensembles ouverts, de savoir que $\overline{G}=0$ pour un seul point R.

6. Supposons que l'ensemble des points extérieurement irréguliers est identique à l'ensemble des points intérieurement irréguliers. Alors les familles F_e et F_i, (voir § 4) coïncident, de sorte qu'on a $\overset{\circ}{\bar{u}} = \overset{\circ}{\underline{u}}$ pour toute fonction u sous-harmonique. Il en résulte que les distributions $\bar{\mu}^P$ et $\underline{\mu}^P$ ont partout même potentiel; donc $\bar{\mu}^P = \underline{\mu}^P$. Il s'en suit alors que la solution extérieure est identique à la solution intérieure pour toute fonction Φ continue.

Puisque tout point intérieurement irrégulier est extérieurement irrégulier, pour qu'on ait toujours $\bar{u} = \underline{u}$, il faut et il suffit donc que tout point irrégulier extérieurement est intérieurement irrégulier. En conséquence de la relation avec la notion de „ensemble effilé en Q" on peut exprimer ceci comme il suit:

Pour que la solution extérieure coïncide avec la solution intérieure pour tout Φ continu, il faut et il suffit qu'il existe une fonction sous-harmonique $u_0(P)$ telle que

$$u_0(Q) > \overline{\lim_{P \to Q}} u_0(P) \qquad (P \subset CE)$$

au voisinage de tout point-frontière Q, n'appartenant pas à E, où il existe pour tout $F \subset CE$ une fonction, sous-harmonique au voisinage de Q, telle que

$$u(Q) > \overline{\lim_{P \to Q}} u(P) \qquad (P \subset F)$$

§ 6. *Le cas d'ensembles mesurables.*

1. Nous supposons pour ce qui suit que *l'ensemble E est mesurable* au sens de LEBESQUE. M. BRELOT a introduit la notion de „distribution-mesure" de E[1]). Il entend par cela une distribution de masses sur E identique à la mesure; le potentiel correspondant est appelé le „potentiel-mesure". C'est une fonction partout continue. Il a alors utilisé ces notions pour un critére de régularité et stabilité aux cas d'ensembles ouverts ou fermés. Ce critère se généralise facilement de sorte que nous renverrons pour la démonstration au mémoire cité. Soit v_E le potentiel-mesure de E. Pour que le point Q de E soit extérieurement (intérieurement) stable il faut et il suffit que la solution extérieure (intérieure) pour E et valeurs-frontière v_E est égale à v_E en Q. Pour que Q soit extérieurement (intérieurement) régulier il faut et suffit que cette solution tend vers $v_E(Q)$ quand P de E tend vers Q.

Remarquons que les points extérieurement (intérieurement) instables sont ceux où $v_E > \overset{\circ}{\bar{v}}_E$ $(v_E > \overset{\circ}{\underline{v}}_E)$. Les points extérieurement (intérieurement) irréguliers sont ceux où $\underline{\lim}\, \overset{\circ}{\bar{v}}_E < v_E$, $(\underline{\lim}\, \overset{\circ}{\underline{v}}_E < v_E)$.

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[1]) M. BRELOT, Critères de régularité et de stabilité. Acad. royale de Belgique, Bull. cl. des Sciences **25** (1939).

En suivant une voie indiquée par M. BRELOT (l. c.) nous déduirons de ce critère une propriété importante concernant les distributions $\overline{\mu}$ et $\underline{\mu}$. Considérons la fonction sur-harmonique

$$w_E = \widehat{(\overset{\circ}{v}_E, v_E)}.$$

On a alors

$$\overset{\circ}{w}_E = \int_{E^*} w_E \, d\overline{\mu}^P,$$

et

$$\overset{\circ}{v}_E = \int_{E^*} v_E \, d\overline{\mu}^P.$$

Or, $\overset{\circ}{w}_E$ prolongé par v_E est sur-harmonique sauf, éventuellement aux points extérieurement irréguliers, puisque hors $\overline{E}$ et aux points extérieurement réguliers $w_E = v_E$. Il suit donc des propriétés des extrémales que $\overset{\circ}{v}_E = \overset{\circ}{w}_E$. Donc

$$\int_{E^*} v_E \, d\overline{\mu}^P = \int_{E^*} w_E \, d\overline{\mu}^P.$$

Il en résulte que $\overline{\mu}^P$ ne peut charger l'ensemble des points de E^* où $v_E > w_E$. En conséquence du critère précédent nous avons donc la propriété suivante:

La distribution $\overline{\mu}^P$ ne charge pas l'ensemble des points instables et irréguliers extérieurement.

De même:

La distribution $\underline{\mu}$ ne charge pas l'ensemble des points instables et irréguliers intérieurement.

Il résulte de la formule donnée ci-dessus pour l'extrémisation d'une distribution quelconque de masses négatives que cette propriété reste vraie au cas général.

Le théorème d'unicité suivant est une conséquence immédiate de cette propriété:

La distribution $\overline{\mu}$ ($\underline{\mu}$) est la seule distribution sur E^*_{esr} (E^*_{isr}) qui conserve le potentiel u de la distribution donnée θ hors $\overline{E}$ et sur E^*_{esr} (E^*_{isr}).

Soit en effet ν une seconde distribution satisfaisant aux conditions imposées. Il résulte de ce qui précède que le potentiel de ν vaut sur E son extrémale. Celle-ci ne dépendant que des valeurs du potentiel sur E^*_{esr}, donc des valeurs de u, le potentiel de ν vaut $\overset{\circ}{u}$ sur E. Il s'en suit que $\overline{\mu} = \nu$.

Dans § 5 nous n'avons pas donné une véritable définition des distributions fondamentales $\overline{\mu}^P$ et $\underline{\mu}^P$, car la manière dont les avons introduites

suppose qu'on connait déjà les solutions extérieures et intérieures (au moins pour certains valeurs-frontière). Les formules données plus tard pour les potentiels de μ^P et $\underset{\sim}{\mu}^P$ ne donnent pas non plus une définition exacte, car pour déterminer les masses correspondants aux fonctions sous-harmoniques $(\overset{\circ}{\overset{\frown}{\bar{u}}},u)$ et $(\overset{\circ}{\overset{\frown}{\underset{\sim}{u}}},u)$, il faut connaître deux extrémales. Or, on voit que la propriété que nous venons de montrer nous donne une véritable définition, indépendante des solutions extérieures et intérieures.

2. Signalons les deux propriétés suivantes, généralisations immédiates de propriétés données par M. BRELOT pour ensembles fermés:

Si E n'a pas d'intérieur l'ensemble des points instables et des points irréguliers extérieurement (intérieurement) est vide ou de mesure positive.

Au voisinage d'un point instable ou irrégulier extérieurement (intérieurement), l'ensemble E^*_{esr} (E^*_{isr}) est de capacité positive.

Dordrecht, février 1940.

Biochemistry. — *Tissues of prismatic cells containing Biocolloids.* I. By H. G. BUNGENBERG DE JONG, B. KOK and D. R. KREGER. (Communicated by Prof. J. VAN DER HOEVE.)

(Communicated at the meeting of March 30, 1940.)

1. *Introduction.*

In a previous communication [1]) the method was described for the inclusion of biocolloidsols in collodion membranes, together with some examples of coacervation of the biocolloids present in the microscopic cavities of the membrane. The method described has a number of drawbacks, however.

a. The membranes are fairly thick, with the result that the cavities in them lie at different levels. The microscopic view is thereby troubled, since these cavities are often seen lying above each other.

b. The preparation requires a fairly long drying-period; if dried too quickly the membrane will not be sufficiently coagulated and the walls of the cavities crack when immersed in water.

c. Before the membrane has at length become sufficiently firm, the watery inclusions sink, the larger ones more especially, owing to their higher specific gravity. If the object-glass is lying with the membrane turned upwards, this is liable to lead to perforation of the membrane on the side touching the surface of the glass. A better way is to dry it with the membrane turned downwards. This perforation is not so liable to take place where air and membrane meet. The membrane separating is so thin, however, that it yet often cracks in contact with the water.

d. A further consequence of the lengthy drying period is the loss of water by the embedded hydrosols. When the dried membrane comes into contact with water, a certain over-pressure is developed in the cavities, which likewise conduces to the large cavities that should be kept intact for the microscopical examination, being injured and the colloids washed away.

Recently a method was found out which is a great improvement upon the method earlier described. It consists herein that the emulsion is not poured out upon an object-glass, but is allowed to spread out upon the surface air/water. The above-mentioned drawbacks are hereby wholly, or for the greater part, eliminated:

a. the preparation takes but a few seconds,

b. the membranes are thin, and the inclusions nearly all lie in the same plane,

c. although some of the larger inclusions may also be injured, the greater number will be found to have the membranes intact.

For other particulars regarding these, see sub 3.

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[1]) H. G. BUNGENBERG DE JONG and O. BANK, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **42**, 83 (1939).

2. *Preparation of the tissues.*

In principle the preparation of the emulsion to be poured on the water surface has not changed in the course of time, except that we have substituted for the ordinary collodion celloidin (SCHERING–KAHLBAUM, für allgemeine Zwecke). Half a tablet of celloidin, cut into small dice in a stopper-bottle, is drenched with 400 cc amylalcohol + 400 cc ether. The celloidin swells and, with occasional shaking, gradually dissolves (in 24 hours). The emulsionizing fluid thus obtained has been used by us for over a year, and will probably keep good still longer.

To prepare the emulsion, 2 cc of the hydrosol to be embedded is put into a measuring-cylinder of 10 cc (preferably with a ground-glass stopper) to which 4 cc of the emulsionizing fluid is added. The emulsification is brought about by turning the measuring glass upside down several times. Violent shaking must be avoided, as otherwise the inclusions will turn out very small.

Then one drop of the emulsion (or several drops quickly one after the other) is allowed to fall from a slight distance on the centre of a Petri dish (diameter 15 cm) filled with distilled water, on the bottom of which an object-glass [1]) has previously been laid. The drop of emulsion will quickly spread out to a thin membrane, in which the inclusions cluster together in small groups. When the membrane has been prepared it begins to shrink in extent. After a few seconds, when this shrinkage is clearly in progress, the object-glass is lifted out of the Petri dish, first tilted sideways, then, still in a slanting position, straight upwards.

The membrane in this way is obtained spread out evenly on the object-glass. It is allowed to drip off for a few seconds longer, and then the object-glass is laid with the membrane side downwards on one of those special cuvettes, to be described later, as are used in microscopic examinations.

Remarks.

a. The emulsion is relatively stable. Although the emulsified drops sink, yet for the greater part they do not run into each other. Probably there will be a film on the boundary surface of the emulsified drops. This has no doubt to do with the fact that membranes obtained with freshly prepared emulsion are often less good than when the emulsion has stood for a short time closed. Apparently it does not take long before the film round the emulsified droplets is sufficiently firm. The keeping properties for yielding good membranes last at least one day, often longer. The emulsion may then be shaken up, and, if the result is not good, a few drops of ether may be added to it. In the case of emulsions containing gelatine, it may be advisable to warm up older emulsions slightly (by holding the tube for 10 seconds in water at 60°).

b. In dropping in the emulsion an extremely thin (monomolecular?) layer spreads rapidly out as far as the sides of the Petri dish, which sometimes hinders a proper

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[1]) Instead of the usual object-glasses we use somewhat wider ones (76 × 32 mm). Hereby it is possible to make use of cuvettes (§ 5) which allow a sufficiently broad strip (15 mm) of membrane for the microscopic examination, and yet have a broad ebonite frame which confines it better.

spreading of the emulsion itself. This can be prevented by running the finger quickly round the rim of the dish directly after the drop of emulsion has fallen on the surface of the water. Good films can also be obtained without this manipulation, however, if the emulsionizing tube be held horizontally for a few seconds close above the surface of the water before letting the drop fall. The ether-vapour over the water-surface will have a good effect here. Or a few drops of ether may be dropped upon the water, and the emulsion poured on when the rippling of the surface has ceased.

c. It will not be necessary to renew the water in the dish before preparing a new tissue membrane. Water which has been used several times for this purpose yields even better membranes than freshly distilled water, as is comprehensible from the effect of the ether upon the spreading. There are, however, some impurities which act unfavourably upon the quality of the membrane, although they do not interfere much with the actual spreading. If, for instance, a little grease be added to the emulsion, fatty acid, paraffine oil (and many other organic compounds), the membrane will lose its cohesive properties more or less after the coagulation.

For such deleterious effects, however, noticeable quantities would have to be used, and they need not be feared from the slight greasy impurities found on ordinary clean laboratory glasses or on the surface of the water/air from the dipped in finger.

3. *The distribution of the inclusions in the celloidin membrane.*

As already mentioned in the introduction, the inclusions in the extremely thin tissue are found as a rule in one and the same plane.

As regards the distribution of the inclusions in the tissue it is possible to modify it oneself. It appears that the volume ratio of the original emulsion determines the structure of the membrane.

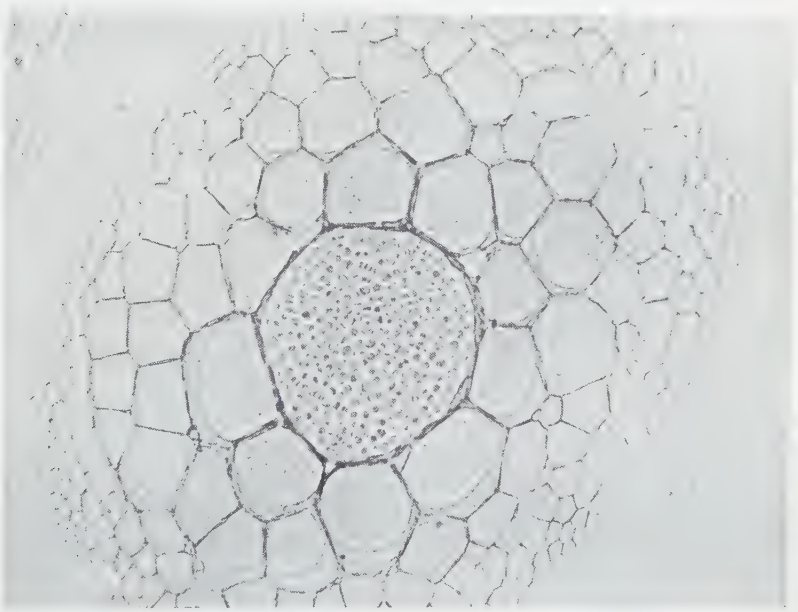
If in preparing the original emulsion, with the same volume of bio-colloidsol much more celloidin solution be taken than given above (sub 2), then the inclusion will lie mainly solitary. Through the microscope they will then appear round, and considering the thinness of the tissue their shape is almost an oblate spheroid.

The proportions of hydrosol and celloidin solution given above, however, result in membranes whereby the inclusions will lie close together in little clusters (e.g. of 20 or more). (See microphoto *A*). The shape of the inclusion is now that of a flat prism, and the majority are 5 or 6-sided prismata. (See microphoto *B*). These clusters are formed during the spreading, but the process is so rapid that the movement of the inclusions in the still fluid membrane towards each other is not perceptible. With respect to the distribution of the individual cells in the cell-groups, we may remark that the largest cells are found in the middle; round about these are the smaller ones, while the very smallest are found at the outside edge. Likewise the cells decrease in height from the centre to the periphery (see also microphoto *B*), so that in principle a cell group in section will have the aspect as shown in Figure 1.

The film is here supposed to be on the underside of the object glass. The thickness of the membranes and the height of the cells has been considerably exaggerated in the drawing. If an object-glass with a membrane be examined at the place where the membrane is bend round the edge of the slide, it will be seen that the membrane over each cell is more or

A 93 ×

B 296 ×



C 134 ×

less convex. This convexity, also to be seen in Fig. 1, is not always equally clear, however.

By carefully focussing the microscope a rather faint circle inside the cell, the larger ones at least, can be observed. This is probably situated in the above mentioned curved membranes of the cell. Respecting the significance and the reason of this particular, we are uncertain. So much we know, however, it is not a circular perforation of the cell membrane, but a thin spot.

Slight disturbances in the structure of a cell-group (as shown in figure 1) are not infrequent. Thus, small cells can be seen pressed in between larger

Fig. 1.

ones, and these are often pushed then to one side of the tissue, whereby the beginning of a three-dimensional tissue-structure is seen (See microphoto *A*).

If a larger plate be used to spread the membrane, and more drops of the emulsion be added on the surface of the water, larger films will be obtained, but the arrangement of the cell-groups and the distribution of the cells in the group is but little different. If the area of the water surface is held constant, but more emulsion be added on the water surface, the cell-groups will be larger.

If less than the above quantity of emulsifying medium be used, the membrane loses in cohesion.

It would seem that then a volume ratio is reached readily resulting in an emulsion of the reverse type (oil in water). With the proportions as given above this is far from likely to occur. However in a tissue there is a cell-group here and there in which a tiny round, spotted brownish ball is seen, which has originated because here a droplet of the emulsion medium happened to get completely embedded in an emulsified drop of the aqueous system.

4. *Permeability of the celloidin walls for biocolloids.*

It is possible in the above-described manner to embed all kinds of typical hydrophile biocolloids. But not all of them yield membranes serviceable for the purposes which we have in view. A requirement is that, in contact with water, the colloids will not gradually diffuse outwards. The method is fairly successful with gelatine, gum Arabic, and mixtures of the two. On the other hand, Na-yeast-nucleinate, in a less degree Na-thymus-mucleinate and K-Chondroitinsulphate, appear to wash readily out of the cells into the surrounding aqueous milieu. Up till now we have not managed, by adding any substance to the emulsion medium, to render the walls of the cells less permeable. Lecithin, cholesterol, fats, fatty acids, camphor, etc., merely led to the membranes losing their cohesive properties after being prepared.

5. *Cuvettes.*

a. Simple cuvette.

For studying the influence of the medium upon the biocolloids embedded in the cells, a simple cuvette may be employed (fig. 2), which consists of a rectangular ebonite frame a few mm in height, cemented on to a glass sheet. Thin glass tubes through the short sides of the ebonite frame are

Fig. 2.

connected via flexible thin rubber tubes (diameter 3 mm) with a reservoir for the fluid on a higher level and a glass tube on a lower level through which the fluid could drop away freely. The cuvette is used in the following way: The upper edge of the ebonite frame is dried with a cloth, and distilled water out of the reservoir is allowed to flow slowly into the cuvette till it is dropping regularly out through the outlet. Then the outlet is closed by a clip, and water is allowed to flow from the reservoir till the cuvette is quite full, and the fluid shows a good head, but does not spread over the ebonite. Then the supply is also cut off with a clip, a membrane prepared, and the object-glass laid on the top of the water with the membrane downwards, whereby a little of the redundant water escapes between the object-glass and the ebonite frame, and the object-glass closes upon the cuvette like a lid. Now the outlet is opened in consequence of which the object-glass is sucked firmly down on to the ebonite frame. The outlet tube must be only so much lower that there is sufficient suction, but no air sucked in through the capillary slit between the ebonite and object-glass. When the supply is then opened, water or any fluid as desired can be allowed to flow past the tissue. Care must only be taken that this flow is not too rapid, as otherwise there is risk of the fluid leaking out through the capillary slit between object-glass and ebonite frame.

It is an advantage to take the glass foundation plate somewhat larger, and to cement round it a second rectangular ebonite frame, so that a space of about 1 cm is left between it and the cuvette proper. In this space the water can collect that is forced out when the object-glass is laid on it, as otherwise the object-table of the microscope is liable to be spilt upon.

When gelatine, or a mixture of gelatine and gum Arabic, is embedded, it is necessary, in studying the coacervation phenomena to work at a higher temperature (e.g. 35°—40°). In such a case the cuvette may be placed on an object-table that can be heated, and the fluid to be flowed continually past the membrane should also be heated beforehand.

b. Cuvette for the study of the influence of the electric field.

The cuvette for this purpose (fig. 3) likewise consists of a rectangular ebonite frame which in this case, however, is thicker (e.g. 1 cm). This is again cemented on to a glass foundation-plate, while, moreover, two blocks of plate-glass (dotted in fig. 3) are cemented into the cuvette, so that a space of only 1 mm is left beneath the object-glass with the tissue.

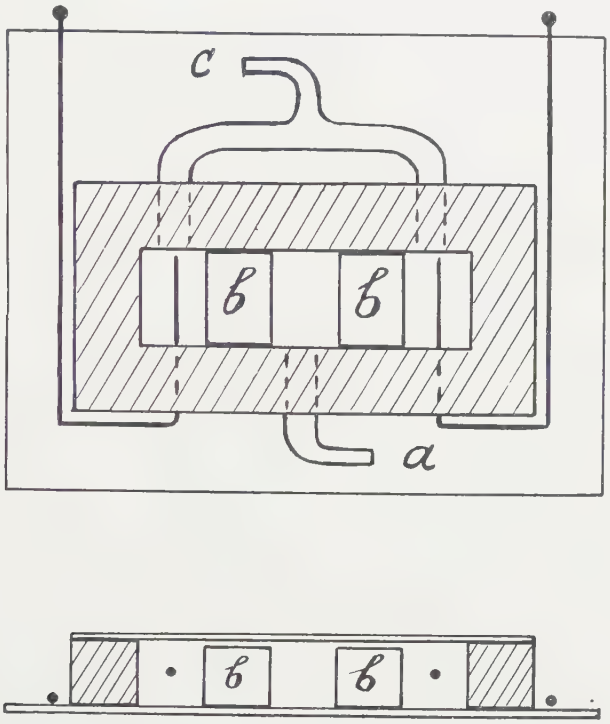


Fig. 3.

This cuvette is employed under a constant flow of the medium, which serves to wash away the electrolyse products on the electrodes. The fluid enters at *a* into the through-like space between the glass-blocks (*b*), flows then over the glass-blocks to the two trough-like spaces in which are the two electrodes (Pt, Ag or even Cu wires), and from thence leaves the cuvette at *c*. The opening *c* is purposely made in the middle of the connecting pipe, so that the resistance of the fluid may be equally great each way along right and left. The glass-blocks (*b*) assist in increasing considerably the speed of the fluid flowing past the membrane, without affecting the economy in the consumption of the fluid. Moreover, this prevents convexion currents in the cuvette, whereby electrolyse products would get to the cell-groups. The proper functioning of the flushing system can be controlled with a quartz suspension, and the off-flow of the fluid can be regulated by placing in the supply-pipe a capillary of such

dimensions that, with the intensity of field used, the rate of the ions with the highest velocity in the electric field (H ions) is exceeded several times.

With the cuvette which we employed, namely glass-blocks 1 cm broad and 1 cm long, with a space of 1 mm between glass-block and tissue, we used only 5 cc flushing fluid per minute, and calculated for a potential difference of 60 volts (i.e. about 30 volts per cm over the glass-blocks [1]) a safety margin about fivefold.

Just as in the case of the former cuvette, it is advisable here, with a view to avoid spilling upon the object-table of the microscope, to fix a second rectangular ebonite frame on to the glass-plate at some distance round the cuvette proper. This is not shown in figure 3.

6. *Complex coacervation of gum Arabic with gelatine in the cells of the tissue.*

The apparatus described is particularly suited to study the coacervation of biocolloids and the influence of a different milieu (pH, salts, etc.), or of an electric field applied to coacervaties already produced respectively. Here, and in the following paragraphs, we shall confine ourselves to describing the formation of a coacervate and the final position of the coacervate in the cell compartment.

If we embed a mixture of gum Arabic and gelatine (6 gr gum Arabic + 5 gr gelatine dissolved in 200 gr water) in the membrane, and allow to flow over the membrane first warm (40°) distilled water, and then for a short time warm (40°) 0.01 N acetic acid, complex coacervation will occur. The way in which this takes place (Cf. fig. 4*A*) is that in each cell very tiny coacervate droplets are separated which perform a distinct Brownian movement. These droplets gradually coalesce to larger ones and these sag in the cell, and as soon as they come in contact with the wall, they lose their outline: the coacervate moistens the wall and gradually the final state can be observed to occur, whereby the coacervate, that still has a few vacuoles, is entirely parietal. This final stadium is more readily obtained in smaller cells than in the larger ones. Compare microphotography C, where in the large middle cell the coacervation is in an incipient stadium (many small coacervate droplets). In the smaller cells lying round about there are still some separate coacervate droplets present, but a considerable number has already moved off to the wall, and forms a parietal coacervate coating. In the cells lying still further outwards the coacervate has already become completely parietal.

7. *The three ways in which the final stadium may be attained, whereby the coacervate is parietal.*

Coacervation does not invariably take place in the manner described

[1]) The field collects practically wholly in the narrow spaces above the glass-blocks. Thus, for studying the behaviour of the biocolloid-system embedded in the tissue (e.g. coacervates), the membrane is examined above the glass-blocks.

above. As a matter of fact three ways are known to us by which the final stadium may be reached (Cf. fig. 4). These are:

a. The coacervate separates as small drops which increase in number and size (partly by coalescing as described above).

b. In the cell compartment tiny vacuoles occur, which increase in number and size, and which finally coalesce with each other to one large central vacuole, where the coacervate, which still holds smaller vacuoles, is also parietal in the final stadium (Fig. 4*B*).

c. In the cell compartment fibres and strands appear which meet in the middle at one or more points. The liberated mass contracts at these points

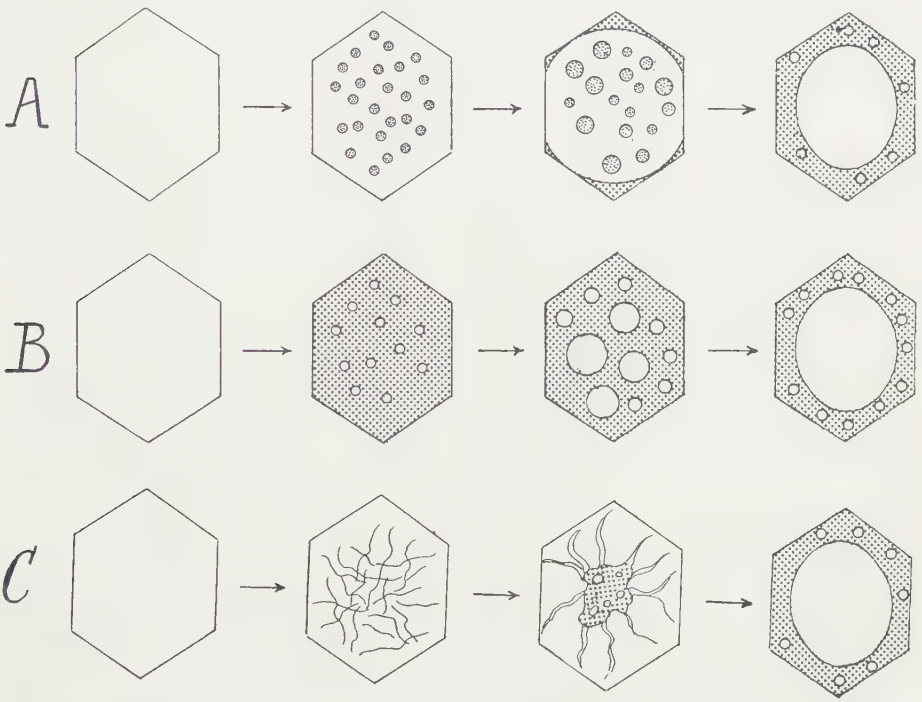


Fig. 4.

to a more or less rounded off and vacuolated clump. This gradually becomes fluid, after which it loses the above mentioned aspect and takes on that of a parietal coacervate (fig. 4*C*).

The way described sub *b* is followed when, proceeding from the same membranes containing gum Arabic + gelatine, we take for a flushing liquid medium first warm (40°) 0.01 N, acetic acid + 40 m. aeq. NaCl, when no changes take place yet, and then 0.01 N acetic acid + 20 m. aeq. NaCl, when the parietal coacervate is brought about via vacuolisation. The explanation is as follows:

The watercontent of complex coacervates is augmented by neutral salts, and the coacervation is suppressed in the case of sufficiently high concentrations. For our combination this already takes place with 40 m. aeq.

NaCl. When we reduce the NaCl, concentration to 20 m. aeq. coacervation is possible. We thus pass a certain salt concentration, whereby a coacervate of high water content i.e. an extremely voluminous, coacervate occurs which fills the whole cell compartment.

With a further reduction of the salt concentration this coacervate becomes poor in water, and the expelled water is liberated in the form of vacuoles. Since, from the nature of the case, the parietal coacervate is not produced by the spreading of coacervate droplets upon the cell-wall, but by the coalescing of the vacuoles, this parietal coacervate still contains, as a rule, a considerable quantity of smaller vacuoles.

The manner described sub *c* is seen when a membrane is cast from an emulsion that has stood a long time cold and this is first flushed with cold 0.01 N acetic acid (when nothing will yet be seen) and then the temperature of the 0.01 N acetic acid be slowly raised to 30—35°. The explanation is that the gelatine in the cold emulsion is present in a gelatinous state. It is true that a complex gel will be produced by allowing cold 0.01 N acetic acid to flow over it, but this, with the microscopic size of our cells, renders the aspect but little different; scarcely anything is seen to take place in the cells.

With slow heating, a gradual transition from the gelatinous state of the gelatine to the sol stadium takes place, whereby the complex gel condenses to a complex coacervate. With the half-melted plastic intermediate stadia there form the coalescing stadia described sub *c*) attached to the wall by strands, while when the transition from gel to sol is complete the normal figure of the parietal coacervate results as the final stadium.

8. *The relative position of coacervate and vacuole.*

In the foregoing it has already been discussed that the final stadium is invariably that whereby in the cell compartment the coacervate is parietal. The remaining cavity, giving rise to the three ways (mentioned sub 7), we have designated as "vacuole".

We are, however, not absolutely certain that this fluid cavity may indeed be termed a vacuole. This would be the case when this fluid hollow is separated on all sides from the cell-wall by coacervate, however thin this coacervate layer might be. If the microphoto C be examined, we get at first sight the impression that the coacervate in the larger cells at least does not entirely surround the liquid cavity. We will now, however, argue that it must not yet be concluded from this that the liquid cavity is not a vacuole in the strictest sense. Proceeding from the assumption that this cavity is indeed surrounded on all sides by coacervate, it then follows from what was discussed in paragraph 3 that, for the smaller cell group illustrated in microphoto C, just a relative position of the vacuole and main mass of the coacervate may be expected in the smaller cells grouped round the larger central cel, as this microphoto clearly shows. We always see there

that the main mass of the coacervate lies turned from the centre of the cell group.

Now, according to 3 the section of a cell group resembles that of a plane-convex lens, in which the large cells lying in the middle are also of the greatest height. This height decreases towards the periphery and, especially when a cell group is fairly small in the case of cells lying round the centrum, the lateral edges turned towards the centre of the group will be longer than the lateral edges turned towards the periphery.

If now, in such a more peripheral cell a certain portion of the volume is taken up by the coacervate, and this completely moistens the cell-wall, the vacuole will seek that position whereby the surface vacuole/coacervate will be as small as possible.

Of the three positions drawn in fig. 5, this surface will decrease in the direction $a \rightarrow b \rightarrow c$. From the point of view that the free surface energy strives to a minimum, we may thus expect just what is seen in the micro-

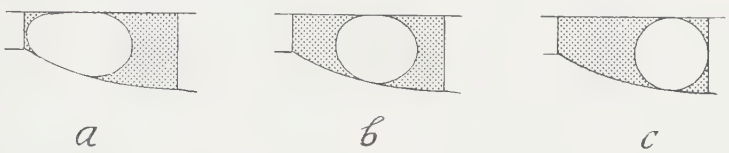


Fig. 5.

photo C viz. in the cells round about the large central cell, the main mass of the coacervate lies turned away from the centrum of the cell-group.

9. *Lesions of the dividing-walls.*

We consider it probable that the coacervate moistens the cell-wall entirely, i.e. that the liquid cavity is a vacuole in the strictest sense.

Nevertheless cells are frequently met with which look as if there is after all a terminal contact/angle coacervate/wall. This aspect is caused, however, by a lesion in the cell-wall there. It can be seen, namely, that in the contiguous cell the coacervate also stands with a terminal angle upon the cell-wall, and just at the same place in the cell-wall. The two cells communicate here, and have only one common vacuole.

In the microphoto C, we see, for instance, in the left lower quandrant three medium-sized cells, lying perpendicularly one above the other, the partition-walls of which have been perforated and which, thus, have one common vacuole.

If coacervation is brought about in the way described in § 6, then we shall also see here and there in the tissue that coacervate droplets flow from the one cell to the contiguous one, proving that the cell-wall here has been injured. If the final stadium of the partietal coacervate be awaited we shall find in these cells the morphological peculiarities which were described above.

Laboratory for Medical Chemistry at Leiden.

March 1940.

Anatomy. — *Ueber Olivenheterotopie.* Von G. M. BAGGERMAN. (Communicated by Prof. H. M. DE BURLET.)

(Communicated at the meeting of March 30, 1940.)

Einleitung.

In der Schnittserie eines menschlichen Hirnstammes wurde als zufälliger Befund eine ungewöhnliche graue Kernmasse angetroffen, welche im Bau der Olive ähnlich sah. Herr Prof. ARIËNS KAPPERS, welcher so freundlich war sich die Präparate anzusehen, erkannte sogleich die ungewöhnliche Lage des abnormen Kernes und regte uns an, den Fall genauer zu untersuchen. Für sein Interesse möchten wir ihn bestens danken.

Leider verfügen wir über keine weiteren Daten, was der Herkunft der betreffenden Serie anbelangt. Von dem dazugehörigen Gehirn ist nichts bekannt. Aus der Grösse möchte man schliessen dass es sich um ein jugendliches Individuum handelt, jedenfalls um ein kleines, denn die Dimensionen sind geringer als diejenigen des erwachsenen Hirnstammes.

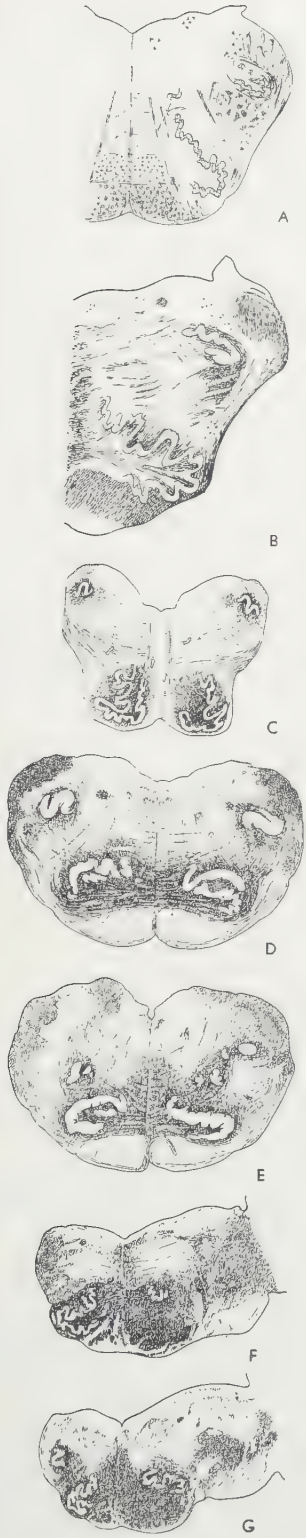
Die Celloidinschnitte (50 μ) sind abwechselnd mit Eisenhämatoxylin auf Markscheiden, mit Karmin auf Zellen gefärbt.

Es zeigte sich bei näherer Untersuchung, dass nicht nur *eine* heterotopische Kernmasse von Olivenkarakter vorhanden war, sondern dass sich deren acht auffinden liessen. Ueber ihre Lage, Grösse, sowie über ihren Bau soll weiter unten berichtet werden.

Vorher soll über einige in der Literatur bisher beschriebenen Fällen kurz berichtet werden. Es ist auffallend dass alle diese Beschreibungen von ebenfalls überzähligen Olivenkernen, sich durch deren Topographie, von unserem Befund unterscheiden, und damit die Darstellung dieses Falles rechtfertigen.

Zum Schluss soll über die Entwicklung des Olivenkomplexen einiges zusammengestellt werden, welches zur Deutung des vorliegenden Befundes beitragen mag.

Aus dem Schrifttum haben wir eine Anzahl Abbildungen, welche zur Erläuterung der beschriebenen Fälle beigegeben sind, zusammengestellt, und in Abb. 1 wiedergegeben. Besonders aus den vier oberen Figuren geht hervor dass die überzählige Olive dorsolateral, im Feld zwischen Corpus restiforme und Tractus descendens Trigemini, angetroffen wird, wie dieses auch aus den Beschreibungen der betreffenden Autoren hervorgeht. Die Abweichung findet sich in den Figuren *C* und *D* zu beiden Seiten, auch bei *A* war das Vorkommen paarig. Figur *E* entstammt derselben Serie wie Fig. *D*, der Schnitt liegt weiter distal und zeigt die überzähligen Kerne



zerstreut im Gebiet zwischen Olive und Corpus restiforme. Die Figuren *F* und *G* haben ebenfalls Bezug auf *ein* Präparat, *G* zeigt links oben an gewöhnlicher Stelle den überzähligen Kern. Die Asymmetrie der Hauptolive in diesen beiden Bildern ist auf die einseitige Rückbildung der Pyramide (in der Figur links) zurückzuführen.

Es handelt sich in diesem, von PRINS beschriebenen Fall, um ein Gehirn mit zahlreichen Abweichungen wobei u.a. der Hirnstamm betroffen war. Dieses gilt übrigens auch für die anderen hier abgebildeten Fälle; stets war die Olivenheterotopie ein Befund, welcher neben anderen pathologischen Zuständen angetroffen wurde. Hieraus könnte man schliessen, dass diese Heterotopie grauer Substanz nur verbunden mit anderen Entwicklungsstörungen des Z.N.S. anzutreffen sei. Es ist aber auch möglich, dass man bei der Untersuchung eines Gehirnes, welches aus irgendwelchen Gründen Interesse bietet, auch den Hirnstamm gründlicher als gewöhnlich untersucht und daher die Möglichkeit der Entdeckung der Olivenheterotopie besonders günstig ist. VON MONAKOW und VOGT haben betont dass auch ohne sonstige Abweichungen eine Olivenheterotopie, für sich alleine, vorkommen kann.

Ueber die einzelnen Befunde sei noch folgendes hinzugefügt: Abb. 1 *A*, entstammt der Beschreibung MEINE's eines hydrozephalen Gehirns mit breiten Gyri welche teilweise verdoppelt waren. Neben dieser Heterotopie grauer Substanz im Gebiet der Rinde, traf Verfasser im Gebiet des Hirnstammes beiderseitig die hier reproduzierte überzählige Olive an. Sie stellt einen medianwärts offenen Beutel dar; seine grösste Länge beträgt 2 mm, seine Breite $1\frac{1}{2}$ mm.

In den zusammenfassenden Arbeiten über

Abb. 1. Heterotopische Olivenkerne nach MEINE (*A*), v. MONAKOW (*B*), VOGT (*C*), MARCHAND (*D* und *E*), PRINS (*F* und *G*).

die Missbildungen des Zentralnervensystems (v. MONAKOW) und über das Wesen mikrocephaler Missbildungen (VOGT) werden die Olivenheterotopien erwähnt und von beiden Autoren mit einer eigenen Beobachtung erläutert (Abb. 1*B* und *C*). Ueber die betreffenden Gehirne wird dort nicht berichtet; ob die Olivenheterotopie auch im Falle v. MONAKOWs (*B*) paarig war, ist dort nicht erwähnt. Zu bemerken ist, dass das Gebilde auch hier einen medianwärts gerichteten Hilus aufweist.

Die Abbildungen *C* und *D* haben Bezug auf den häufig zitierten Fall von MARCHAND. Es handelt sich um ein mikrocephales Gehirn. Seiner Beschreibung des Hirnstammes sind folgende Sätze entnommen: „Die auffallendsten Eigentümlichkeiten zeigen die Oliven; die gefaltete Lamelle derselben ist verhältnismässig dick und weniger regelmässig und zierlich als normal. Von dem hinteren Umfang beider Oliven lösen sich einige unregelmässige rundliche Massen von grauer Substanz ab, welche ganz die Beschaffenheit der normalen Oliben haben; dieselben treten in wechselnder Zahl und Grösse auf allen Querschnitten zwischen Oliven und Corpora restiformia auf. In der mittleren Höhe der Oliven treten diese Massen jederseits zu einer zweiten, aus einem doppelt gefaltenen Bande grauer Substanz bestehenden Nebenolive zusammen, welche medianwärts und vor dem Corpus restiforme gelegen ist. Die beschriebenen Nebenoliven sind keineswegs gleichbedeutend mit den normalen Gebilden gleichen Namens, denn diese sind ausserdem vorhanden. Die ersteren sind nach Art der Hauptoliven mit einem System bogenförmiger Faserzüge umgeben".

Ueber den Fall PRINS' (Fig. *F* und *G*) haben wir oben bereits kurz gesprochen, es sei noch folgendes hinzugefügt. Wie beim Fall MARCHAND handelt es sich um eine Vielheit von überzähligen Nebenoliven. Diese wurden angetroffen in der Medulla eines Gehirnes, welches auch zahlreiche andere Abweichungen aufwies.

Eigene Beobachtung.

Zur vorläufigen allgemeinen Orientierung sei zunächst auf die Abbildung 2 verwiesen. Aus dem Hirnstamm ist ein Stück herausgeschnitten worden wovon nur die Oliven gespart geblieben sind. Man erkennt die Hauptolive (3), die dorsale (2) und mediale (1) Nebenolive, jederseits der mitangegebenen Medianebene. Diese Gebilde setzen sich jenseits der Schnittebene, aus welcher sie in distaler Richtung hervorragen, noch ein Stück proximalwärts fort. Die überzähligen Kerne liegen in zwei Reihen unweit der Medianebene. Rechts sind zwei, links vier vorhanden. Ausser diesen median gelegenen, finden sich noch zwei überzählige Kerne, ebenfalls ungefähr symmetrisch, in lateraler Lage, und zwar im Corpus restiforme. Im ganzen finden sich also acht graue Kernmassen von Olivenkarakter.

Genaueres über ihre Lage lehrt die Abbildung 3, welche als Projektionszeichnung auf die Medianebene einerseits, auf eine senkrecht dazu stehende Frontalebene andererseits ausgeführt wurde und welche auch den normalen

Olivenkomplex enthält. Es geht aus dieser Figur hervor, dass die Gruppe der medialen Kerne (*a, b, c, d, e, f*) etwa in derselben Sagittalebene liegt wie die beiden medialen Nebenoliven. Sie liegen dabei, wie eine Schnur, alle etwa in gleicher Höhe.

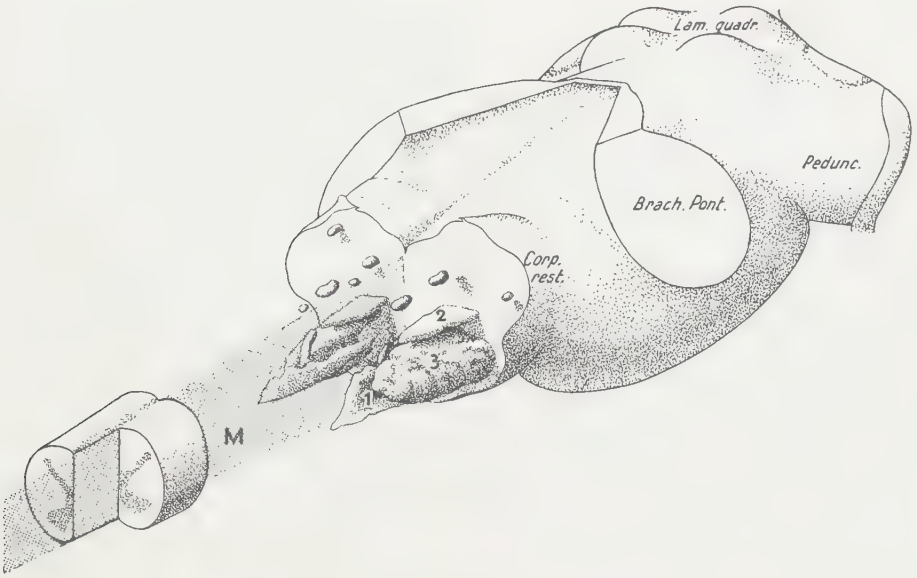


Abb. 2. Hirnstamm von rechts. Der hintere Teil der Medulla oblongata ist entfernt, um die Lage der überzähligen Oliven in Bezug auf die Normalen zu veranschaulichen. 1. 2. mediale und dorsale acc. Olive. 3. Hauptolive. M. Medianebene.

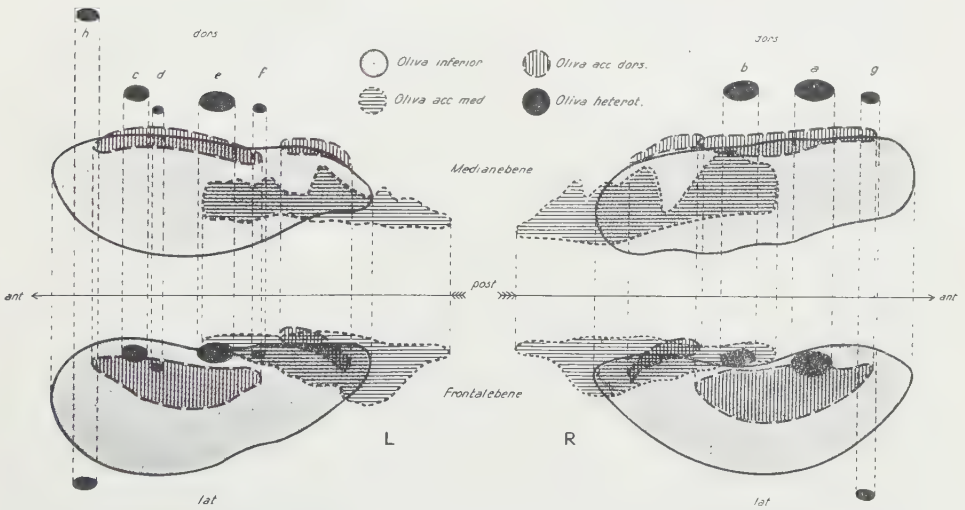


Abb. 3. Projektionszeichnung zur Veranschaulichung der beiderseitigen Lagebeziehungen der heterotopischen Olivenkerne (*a* bis *h*) zum normalen Olivenkomplex. Oben Projektion auf der Medianebene; unten, Projektion auf der Frontalebene.

Die abweichende Lage der lateralen, als *g* und *h*. bezeichneten Kerne, geht besonders aus der Projektion auf der Frontalebene (Abb. 3 unten) hervor. Wie auch die Abbildung 2 zeigt, liegt *h* (links) höher als *g*.

Es sei betont, dass Lage und Grösse der beiderseitigen Haupt- und Nebenoliven, keine erkennbaren Abweichungen aufweisen.

Die Abb. 4 zeigt einen der medialen heterotopischen Kerne im Querschnittsbild, und zwar den Dritten (*e*) aus der linken medialen Reihe. Die Hauptolive und die beiden Nebenoliven sind ebenfalls getroffen und es zeigt sich auch hier, dass der Kern *e* oberhalb des nucl. olivaris accessorius medialis liegt. Schon bei dieser schwachen Vergrösserung erkennt man, dass es sich um ein gebogenes Kernblatt handelt mit einem nach unten gerichteten Hilus. Beachtenswert ist die Tatsache, dass die austretenden Hypoglossusbündel stets an der medialen Seite der überzähligen Kerne gefunden werden.

Von demselben Nebenkern *e*, sowie von dem gegenüberliegenden Kern *a*, sind in Abb. 5 Schnittbilder bei stärkerer Vergrösserung wiedergegeben. Das linke Bild zeigt denselben Kern, welcher in Abb. 4 wiedergegeben wurde. Jedoch hat das Markblatt hier einen geschlängelten *s*-förmigen Umriss. Es ist von einer Kapsel von markhaltigen Nervenfasern umgeben. Der feinere Bau stimmt mit demjenigen der Olive überein; in der Schlängelung des Markblattes zeigt sich die Uebereinstimmung mit dem Hauptkern der Olive.

Der gegenüberliegende Kern *a* (Abb. 5 rechts) weist dieselben Merkmale auf. Die Kapsel ist an dieser Stelle weniger ausgeprägt, aber doch vorhanden; das gebogene Markblatt mit dem nach ventral gerichteten Hilus tritt klar hervor. Es konnte nicht festgestellt werden ob die Fasern aus dem Nebenkern sich, wie diejenigen der Hauptolive, in Richtung des gegenüberliegenden Corpus restiforme begeben.

Von den heterotopischen Kernen *a* bis *f* wurden bei hundertmaliger Vergrösserung Wachsmodelle hergestellt, um einen Eindruck über ihre Form zu erhalten. Von diesen Modellen ist in Abb. 6 eines, und zwar dasjenige von Kern *a*, wiedergegeben. Die Faltung des Markblattes geht

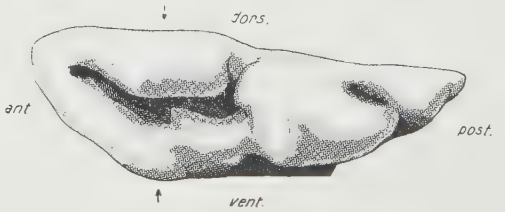


Abb. 6. Wachsmodell des Nebenkernes *a* bei vierzigmaliger Vergrösserung von medial unten gesehen. Die Pfeile entsprechen der Lage des Schnittbildes 5 *R*.

aus derselben deutlich hervor. Der Kern ist von medial unten gesehen und der nach unten gerichtete Hilus ist im vorderen Abschnitt desselben gut ausgeprägt. Die Pfeile geben die Lage des Schnittes 5 *R* an. Es erübrigt sich auf die Gestalt der anderen Nebenkerne einzugehen. Wenn ein Hilus

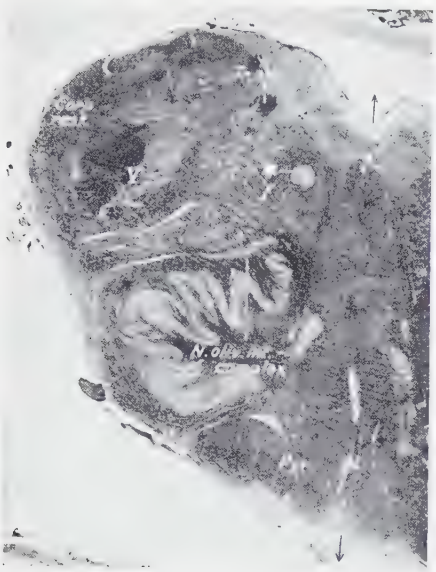


Abb. 4. Caudalansicht der linken Hälfte eines Medullaquerschnittes, welcher neben den nucl. oliv. inferior, den heterotopischen Olivenkern *e* enthält. (Vergr. 6 ×.)

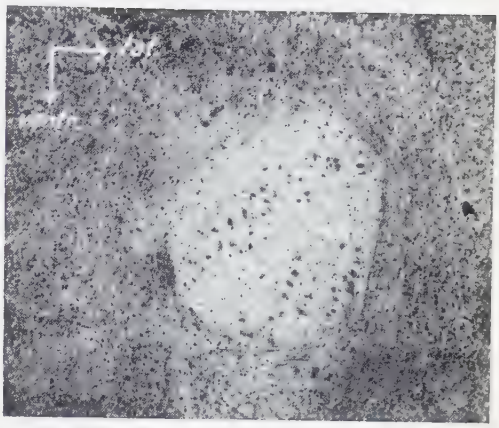


Abb. 8. Der Nebenkern *g* des rechten Corpus restiforme bei stärkerer Vergrösserung. (75 ×.)

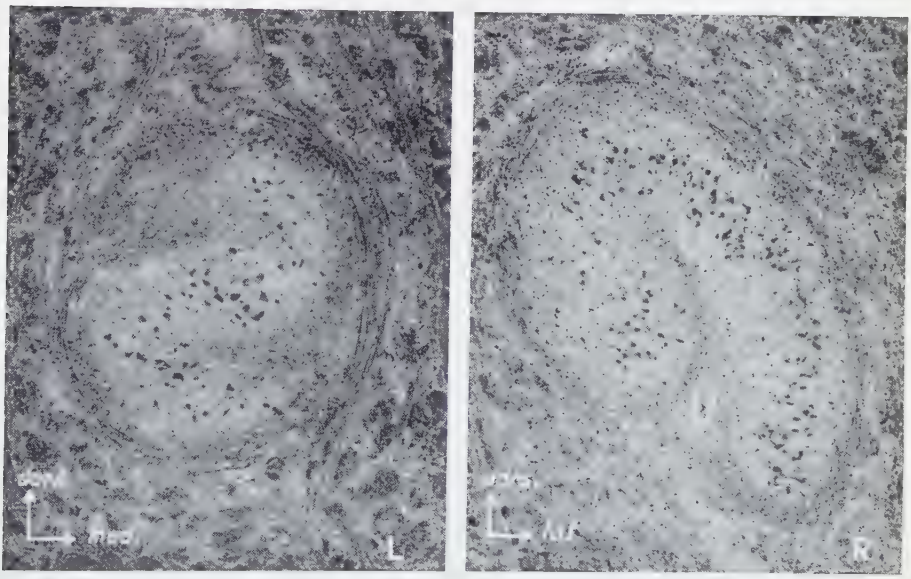


Abb. 5. Zwei heterotopsiche Olivenkerne bei stärkerer Vergrösserung. L. Schnitt durch den Kern *e*; R. Schnitt durch den Kern *a* der Abb. 3. (Vergr. 70 ×.)

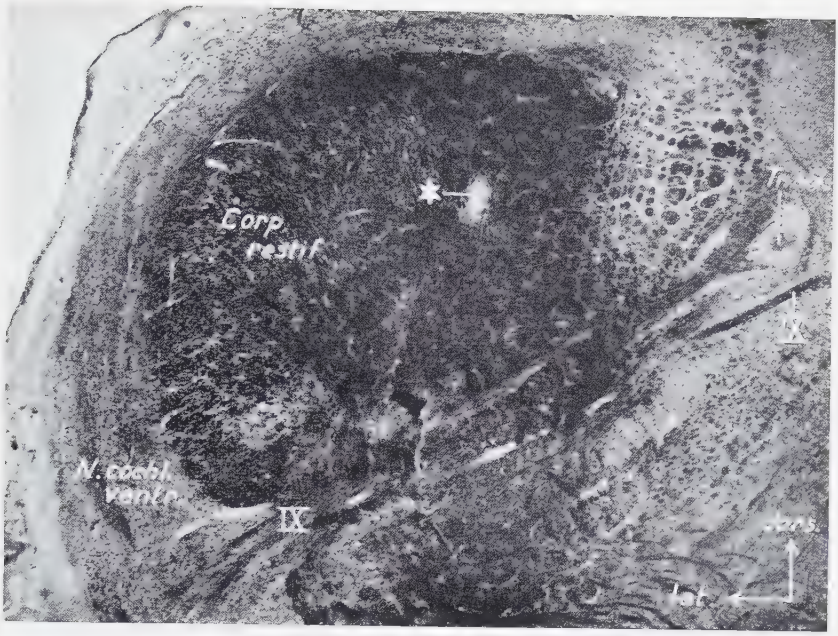


Abb. 7. Uebersichtsbild des linksseitigen Corp. rest. zur Veranschaulichung des heterotopischen Kernes *h*. (Vergr. 16 ×.)

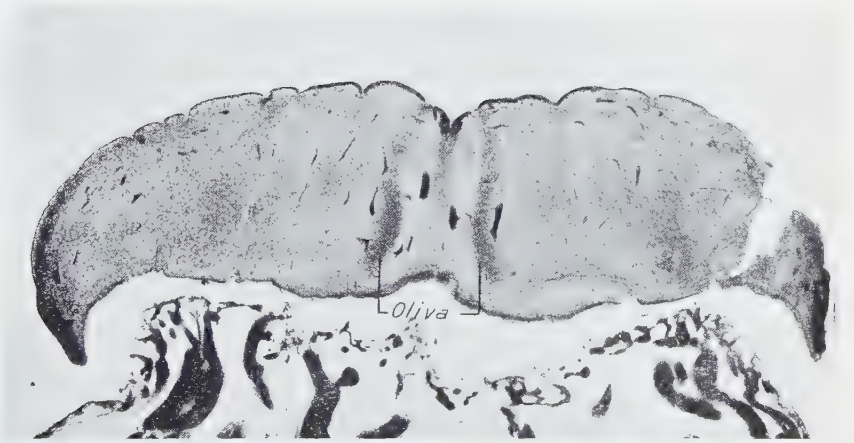


Abb. 11. Querschnitt durch das verlängerte Mark eines Embryo mit Encephalomyeloschisis totalis, welcher die Anlage der (medialen accessorischen?) Olive zeigt.

vorhanden ist, und das ist bei den vier grösseren (*a, b, c, e*) stets der Fall, so ist dieser nach unten oder nach unten medial gerichtet. Nur die beiden kleineren Kerne *d* und *f*, haben keinen Hilus, sie sind Kugelförmig.

Auch die beiden lateral im Corpus restiforme gelegenen Nebenkerne haben keinen Hilus. Der linksseitige ist in Abb. 7 bei schwacher Vergrösserung wiedergegeben. Es ist ein längliches Gebilde, welches sich durch zweiundzwanzig Schnitte erstreckt, und hat also etwa die Länge des medialen Kernes *a*. Der gegenüberliegende Kern *g*, liegt etwas weiter distal und ventral, er ist kürzer und breiter als *h*, und von einer Faserkapsel umgeben. Ein Bild desselben bei stärkerer Vergrösserung, bringt die Abb. 8, auf welcher man den typischen Olivenbau erkennt. Was ihre Topographie anbelangt erinnern die beiden letztgenannten Kerne an die Heterotopien, welche von v. MONAKOW, VOGT, MEINE u.s.w. beschrieben wurden, und welche in Abb. 1 reproduziert wurden. Jedoch ist zu bemerken, dass in allen den zitierten Fällen die Kerne zwischen Corpus restiforme und Tract. spin. descendens trigemini gelegen waren, während sie hier im Corpus restiforme angetroffen wurden.

Besprechung.

Es erhebt sich die Frage ob sich über die Herkunft der überzähligen Olivenkerne, sowie über ihre eigenartige Topographie etwas aussagen lässt.

Die Ansichten über die Entstehung des Olivenkomplexes sind auf HIS zurückzuführen. In seiner Arbeit über die Entwicklung des menschlichen Rautenhirns beschreibt er, wie aus der verdickten Lippe, wo der Boden des Ventrikels in das dünne Dach übergeht, die „Rautenlippe", sowohl das Kleinhirn, wie eine Anzahl von Kernen des Hirnbodens hervorgehen. Es handelt sich hier um die Kerne des Pons, die Nucl. raphe, die Nucl. arciformes, die Oliven, sowie um die Nucl. laterales. Dieses gemeinsame Ursprungsgebiet, die Rautenlippe, ist in Abb. 9, schematisch wiederge-

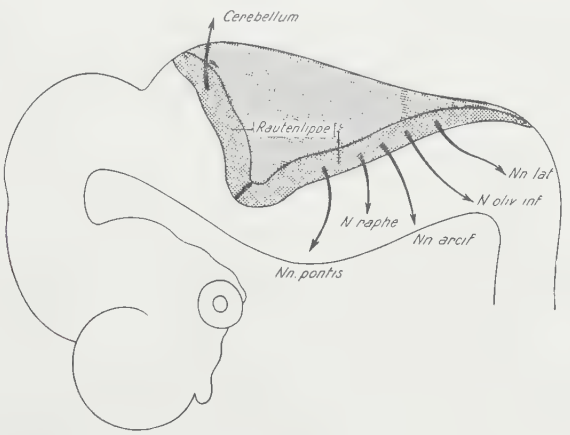


Abb. 9. Umrisszeichnung des Gehirnprofils eines Embryo von 10,2 mm mit Angabe der Derivate der Rautenlippe, abgeändert nach HIS.

geben. Es geht aus derselben hervor, dass „die Rautenlippe sich jederseits erstreckt von der Nackenbeuge bis zum Isthmus. Aus der Lippenplatte gehen alle die Zellen hervor, welche dem Complex der zerrissenen Kerne angehören", unter welchen His die obengenannten Kerne versteht. „Die Oliven und ihre Nebengebilde sind demnach, soweit es sich um ihre Zellen handelt Abkömmlinge der Flügelplatte und morphologisch entstammen sie derselben Längszone des Markrohres, aus der in höher gelegenen Abschnitte des Gehirns, das Cerebellum u.s.w. hervorgehen." In diesen wenigen Sätzen ist das Wesentliche der Auffassung von His mit seinen eigenen Worten wiedergegeben.

Der Zusammenhang der aus dem gemeinsamen Boden der Rautenlippe entstehenden Gebilde, tritt in seiner Verbundenheit auch im klinischen Bilde hervor. Déjérine und Thomas beschrieben erstmalig im Jahre 1900 einen Fall, welchen sie als Atrophia olivo-pontocerebellaris benannten, bei welchem gleichzeitig das Kleinhirn, die Brückenkerne, und die Oliven betroffen waren. Seit dieser Beobachtung sind eine Reihe weiterer klinischer Veröffentlichungen gefolgt, aus welchen sich der Zusammenhang der obengenannten Areale ergibt. (Holmes–Stewart, Brouwer–Coenen, Hoeneveld, Bakker, Prins, Koster, u.s.w.). Es ist hier nicht am Platze auf diese interessanten Arbeiten einzugehen, da sie über die Entwicklung der Olive keine Gesichtspunkte bringen.

Kehren wir zur Entwicklungsgeschichte zurück und fragen uns auf welcher Weise, auf welchem Wege, erreichen die Zellen aus dem Gebiet der Rautenlippe ihre endgültige Bestimmung.

Abb. 10. Querschnittschema des Bodens der Rautengrube mit Angabe der Richtung zweier Zellströme, welche von der Rautenlippe an der ventralen Oberfläche (1) und an der dorsalen Oberfläche sich medialwärts zum Olivengebiet begeben.

Wandern sie dabei an der äusseren Peripherie des Hirnstammes nach abwärts (Abb. 10, Pfeil 1) oder betreten sie auch mehr medial gelegene Pfade (Pfeil 2). Es ist unzweifelhaft, dass die Mehrheit der Zellen sich an der Peripherie entlang verschiebt, jedoch sind in den Abbildungen von His, auch Zellströme angegeben, welche mehr medial verlaufen. Es ist bemerkenswert, dass diese Abbildungen von His mehr als zwanzig Jahre später, in Keibel und Mall's Handbuch, von Streeter im Kapittel über die Entwicklung des Nervensystems aufs Neue verwendet wurden. Um diese Zeit trat ein neuer Bearbeiter dieses Gebietes in den Vordergrund. Dieser Verfasser, C. R. Essick traf als Zufallsbefund an den lateralen Rand der Rautengrube eine Erhebung (Corpus ponto-bulbare) an, deren Bedeutung er zu ermitteln versuchte. Es zeigte sich, dass diese Masse grauer Substanz aufgefasst werden musste als ein Derivat der Rautenlippe, welches seine Abwanderung in die definitiven Kerngebiete nicht, oder unvollständig, zur Ausführung gebracht hatte. Er schliesst sich der Auf-

fassung von His an, dass die meisten Zellen der äusseren Oberfläche entlang ihrem Ziel zustreben. (Pfeil 1, Abb. 10); möchte diesen Weg jedoch nicht als den einzig möglichen anerkennen. U.a. geht das aus folgendem Schlusssatz hervor, „The Rautenlippe of His is the common ancestor of the olive, pontine nuclei and arcuate nuclei"; sie entstehen „along with p a r t s [1]) of the olive by a superficial migration over the ventral surface of the medulla". Das bedeutet, dass ein anderer Teil der Olivenzellen nicht oberflächlich migriert ist, d.h. diese Zellen wanderten wie Pfeil 2, Abb. 10.

Als dritten Autor möchten wir Streeter anführen, welcher in seinem oben zitierten Handbuchbeitrag folgendes zur Entwicklung der Oliven berichtet. „Am äussersten lateralen Rande der Flügelplatte, in der Rautenlippe, finden sich Zellen, welche auch beim Erwachsenen ihr primitives, embryonales Aussehen behalten. Andere dringen in die Randschicht ein und tauchen an der Oberfläche der Wand auf. Von diesen wandern einige an der Oberfläche, peripher von der Randschicht, gegen die Mittellinie hin. Das genaue Schicksal dieser Zellen muss noch untersucht werden. Es ist möglich, dass die weiter caudal gelegenen an der Bildung der nuclei arcuati Anteil haben, möglicherweise auch an der, der Olivenkerne."

Kooy hebt in seiner ausführlichen Arbeit über die vergleichende Anatomie des Olivenkomplexes und ihre Entstehung, auch auf Grund eigener Befunde, den lateralen Migrationsweg besonders hervor. Uebereinstimmend berichten demnach die genannten Autoren dass der Olivenkomplex, aus an der Peripherie des Hirnstammes migrierenden Zellen entsteht; es bleibt aber die Möglichkeit offen, dass andere Elemente Bahnen folgen, welche mehr im Inneren des Hirnstammes gelegen sind. Es ist schliesslich zu erwägen ob nicht lokal, im somatisch motorischen Gebiet vorhandene Zellen am Aufbau der Olive beteiligt sein könnten. Zu dieser letzteren Auffassung veranlasst uns mit ein Schnittbild, welches wir in Abbildung 11 wiedergeben, und welches der Serie eines Embryo mit Encephalo myeloschisis totalis entstammt. Die Olive ist in diesem abweichenden Objekt vertreten durch eine Zellsäule, welche die ganze Tiefe der offenen Medullarplatte von oben nach unten durchsetzt.

Obwohl auch in diesem Objekt im Gebiet der Rautenlippe viele Zellen vorhanden sind, lassen sich keine Zellströme nachweisen, welche diese Region mit der Olivenanlage verbinden. Wir erinnern hier an den oben zitierten Satz von Streeter in welchem ebenfalls Zweifel über die Herkunft der medialen accessorischen Olive aus lateralen Kerngebieten geäussert wurde.

Schliesslich sei hier noch eine Bemerkung Edingers erwähnt. Ueber die accessorischen Olivenkerne äussert er sich folgendermassen: „Es ist möglich, dass sie ventrale Abschnitte des motorischen Haubenkerns darstellen,

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[1]) Von uns gesperrt.

und das die Oliven nur spezielle Differenzierungen der gleichen Kernmasse darstellen".

Wenn demnach EDINGER über die Herkunft wesentlicher Bestandteile der Olivenkerne im unklaren ist, so verdient es doch Beachtung, dass dieser hervorragende Kenner des Zentralnervensystems eine lokale, mediale Entstehungsart der Olive befürwortet.

Wenn wir auch über die Herkunft der hier geschilderten überzähligen Olivenkernen nur Vermutungen äussern können, so geht doch aus obigen Betrachtungen zur Genüge hervor, dass der Annahme einer lokalen Differenzierung derselben wesentlichen Bedenken nicht entgegenstehen. Es würde sich dann um Olivenbestandteile handeln, welche den Anschluss an den Hauptkern verfehlt hätten. Eine weitere Möglichkeit wäre diese, dass Reste eines von lateral nach medial verlaufenden Zellstromes (Pfeil 2, Abb. 10) ihre Endstätte nicht erreicht hätten, um sich nun als überzählige Olivenkerne zu manifestieren. Für letztere Auffassung würde der Befund der beiden lateral im Corpus restiforme gelegenen, überzähligen Olivenkerne sprechen.

Schliesslich wäre zu erwägen aus welchem Grunde diese Kernbestandteile an ihrem ungewöhnlichen Platz verharren, statt sich dem Olivenkomplex bei zu gesellen.

Es erhebt sich dabei die Frage, ob benachbarte Kerne auf neurobiotactischem Wege der normalen Abwanderung dieser Zellgruppen im Wege getreten sein könnten. Wo die überzähligen Kerne über ein so ausgedehntes Gebiet angetroffen werden, wo sie nachbarliche Beziehungen zu verschiedenen anderen Kernen aufweisen, ist es uns nicht gelungen diesbezüglich einen einheitlichen Gesichtspunkt ausfindig zu machen.

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Astronomy. — *Spontaneous development of a gaseous disc revolving round the sun into rings and planets.* I. By H. P. BERLAGE Jr., Research Associate Roy. Magn. and Meteorological Observatory, Batavia.

(Communicated at the meeting of March 30, 1940.)

In 3 previous papers [1] [2] [3] the author developed the theory of a gaseous mass in steady revolution round the sun and showed that it possessed the characteristics of an embryo of the planetary system. Let us recapitulate first what was essential.

If r denotes the distance of a volume element from the axis of rotation, h its height above or below the equatorial plane, p and ϱ the pressure and density of the gas, ω the angular velocity, M the mass of the sun and f the constant of gravitation, the two fundamental equations of equilibrium are

$$\frac{fMh}{(r^2+h^2)^{\frac{3}{2}}} + \frac{1}{\varrho}\frac{\partial p}{\partial h} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{fMr}{(r^2+h^2)^{\frac{3}{2}}} + \frac{1}{\varrho}\frac{\partial p}{\partial r} = \omega^2 r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

When we introduce the gasequation

$$p = \varrho RT. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and assume, as we have always done, for the sake of simplicity

$$RT = constant \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

the integration of the equations of equilibrium leads to

$$\frac{fM}{r^2} + RT\frac{d\,lg\,\varrho_e}{dr} = \omega^2 r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and

$$\varrho = \varrho_e\, e^{-\frac{fM}{RT}\left[\frac{1}{r}-\frac{1}{\sqrt{r^2+h^2}}\right]} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

These equations determine ω and ϱ in function of ϱ_e, the density of the

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[1] On the structure and internal motion of the gaseous disc constituting the original state of the planetary system. Proc. Kon. Akad. v. Wetensch., Amsterdam, **35**, 553—562 (1932).

[2] Viscosity and steady states of the disc constituting the embryo of the planetary system. Proc. Kon. Akad. v. Wetensch., Amsterdam, **37**, 221—232 (1934).

[3] The theorem of minimum loss of energy due to viscosity in steady motion and the origin of the planetary system from a rotating gaseous disc. Proc. Kon. Akad. v. Wetensch., Amsterdam, **38**, 857—862 (1935).

gas in the equatorial plane. The rotation appears to be cylinderwise, ω depending on ϱ_e, which is a function of r only.

The nebula has no finite boundary. It is a whirl in infinite space filled with matter. However, for reasonable values of the product RT the density of the nebula decreases so rapidly in axial direction on both sides of the equatorial plane that it takes the shape of a disc merging into interstellar matter at values of $h \ll r$.

Hence, we are allowed to exchange (6) by

$$\varrho = \varrho_e \, e^{-\frac{fM}{2RT}\frac{h^2}{r^3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

down to the limit, where ϱ reaches values as low as 10^{-24} gr/cm^3. We even venture to calculate the total mass and moment of momentum of the system with the aid of (7) and obtain

$$m = (2\pi)^{\frac{3}{2}} \left(\frac{RT}{fM}\right)^{\frac{1}{2}} \int_0^\infty r^{\frac{5}{2}} \varrho_e \, dr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\theta = (2\pi)^{\frac{3}{2}} \left(\frac{RT}{fM}\right)^{\frac{1}{2}} \int_0^\infty r^{\frac{9}{2}} \omega \, \varrho_e \, dr \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The advantage of this operation is that we skim off the solar nebula itself from an infinite interstellar medium, from which it may have been contracted by the sun, but which is of no importance in the following course of evolution.

When we give ϱ_e in function of r, the structure of the gaseous mass is completely determined. In the second paper I adopted as probable

$$\varrho_e = \varrho_0 \, e^{-ar} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

with ϱ_0 for the density of the gas at the centre. A meridional section through the disc was designed, presenting unexpected features. In the same paper it was shown that the restriction (4), which was introduced to simplify the theory, has a chance to be nearly fulfilled actually. When $\bar{\mu}$ denotes the mean molecular weight of the gas (4) reduces to

$$\frac{T}{\bar{\mu}} = \text{constant} \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

However, T and $\bar{\mu}$ will show both about the same tendency to decrease with increasing distance from the sun. Hence, there is a lack of knowledge of the temperature of the nebula, but no reason to fear that we have in any serious way confined the general applicability of the theory.

In my third paper, with the aid of the theorem of minimum loss of energy due to viscosity in steady motion, I could prove that (10) has to be replaced by

$$\varrho_e = \varrho_0 \, e^{-ar^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The only 2 independent variables, ϱ_0 and a, can be determined by (8) and (9) from the known invariable total mass and total moment of momentum of the planetary system.

It is a famous difficulty, threatening any theory of the origin of the solar system, that the moment of momentum of the planets is more than 20 times bigger than the moment of momentum of the sun. How did the almost total mass of the system concentrate in the central body, whereas the almost total moment of momentum is distributed over the small bodies in revolution round their primary?

The difficulty is evaded when we admit that the sun by its gravitation gathered the cloud, which was to generate planets, from interstellar matter on its course through the Galaxy. In this case, in a first approximation, the moment of momentum of the sun is independent of the moment of momentum of the nebula. When in reality the sun possesses a fraction of the moment of momentum of the nebula, with a slightly different orientation, this need not have any other meaning than that not more than a fraction of the same order of magnitude of the matter has been united with the solar body.

There is, however, one serious objection. If it is true that radioactive atoms can only have been built up in the interior of stars, radioactivity on earth could hardly still exist if this matter of the earth had not constituted part of the sun or some other star up to a recent date of, say, 3 or 4000 million years ago.

If the validity of this objection is granted, the author considers this to be the point, where the Tidal Theory could solve the difficulty. It may be that 3 or 4000 million years ago a foreign star approached and tore from the sun a fraction of its matter and imparted to this matter sufficient moment of momentum round its primary. More than this the Tidal Theory can hardly explain. Indeed, we have to imagine, how the foreign star left the sun and its tail in a most disorderly condition. Stellar encounter or not, the original state in which we find our planetary system is a chaotic nebula revolving round the sun.

This nebula was modelled by viscosity from a structureless cloud into the rotating disc, which the author has described, the shaping advancing at first rapidly but roughly in turbulent motion, later more slowly and slowly while turbulence is gradually dying out and the ordinary viscosity of laminar motion remains the controlling factor. Finally the tendency to minimum loss of energy due to viscosity leaves the disc in a next to stationary motion. Secular stationarity would only be reached if the whirl were enclosed within a cylindrical wall of finite radius and no exchange of matter between sun and nebula were possible. As to the actual situation, we are starting our present investigation with a gaseous disc, the structure of which has been completely determined, rotating cylinderwise round the sun in nearly steady motion. The question is, whether it remains a disc or what else might happen further.

It needs hardly be repeated here that no local accidental concentration of matter in the nebula will ever lead directly to further concentration into globes and planets. Tidal forces will wipe away any concentration before its density has reached the minimum value required by ROCHE for a material body to keep its individuality by its own attraction. However, tidal forces do not prevent concentration of matter into rings concentric with the sun. No more are these rings necessarily instable.

JEANS in his criticism of the theory of LAPLACE [1]) proved that a ring of too low density will expand, but he supposes the ring to rotate approximately as a rigid body. POINCARÉ, however, allowing for a variation of the angular velocity with distance from the axis of rotation, found that a gaseous ring of low density may be stable [2]). It looses its stability when growing denser or when friction has equalized the angular velocity in a critical degree.

The structure of a ring in steady motion follows from the equations of equilibrium for a given distribution of ϱ_e, We need not be effrayed of the conclusion that such a ring has no finite boundary. Not even the atmospheres of the planets in their actual state have one. The individuality of a celestial body depends on a depression of the density of its atmosphere below a certain small value at reasonable distances from the main body.

Let us follow here for a few moments what will probably occur, when a ring is disintegrated through instability. It will be broken up in a few separate clouds pursuing each other in nearly one and the same circular orbit. The angular velocity of these bodies never being strictly equal, they will finally unite. In two other papers [3]) [4]) the author has endeavoured to prove that the resulting nebula will normally be spinning in the "direct" sense round an axis perpendicular to the plane of the orbit, whereas, when the ring has been sufficiently wide the outer portions of the nebula will show a "retrograde" motion. This nebula will produce one planet and its satellites.

The only remains of the ring which do not combine with the planet are those which happen to be with the planet in one of the two well known equilateral positions towards the end of the process of condensation. This explains the origin of the Trojans preceding and following Jupiter. Reversing the argumentation, we may even consider the existence of the Trojan group of planetoids as one of the best empirical proofs of any ring theory.

There was a time, when the authors theory of the electrostatic field of

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[1]) Problems of cosmogony and stellar dynamics, p. 212.

[2]) Leçons sur les hypothèses cosmogoniques, p. 49.

[3]) Versuch einer Entwicklungsgeschichte der Planeten, GERLANDs Beiträge z. Geophysik, 17 (1927), Ergänzungsheft.

[4]) A study of the systems of satellites from the standpoint of the disctheory of the origin of the planetary system. Annalen BOSSCHA Sterrenwacht, Vol. 4, Miscellaneous papers N^0. 7 (1934).

the sun due to its corpuscular rays made him believe that no other factor than this electrostatic field could have induced the condensation of the disc into rings [1]). It was a theory in which these rings could be compared with CHLADNI's figures on a vibrating plate and specially suggestive by the fact that BODE's law followed as a logical consequence. However, doubt was cast upon the adequacy of this theory when it was proved that the satellite systems are in more than one respect true copies of the planetary system. There is no room for any other conclusion than that the creation of the secondary systems has been a repetition of the birth of the primary system on a smaller scale.

If our planets have produced electrostatic fields they should once have emitted charged corpuscles. The emission of charged corpuscles by the sun is a fact, the emission by the planets a hypothesis ad hoc. These and other quantitative difficulties have convinced the author that his theory was an error and not even an elegant one, because he has now found definite proof that the gaseous disc rotating round the sun does not require any external forces, but will spontaneously condense into rings and planets.

Let us follow the way, which was followed by others in comparable cases. When POINCARÉ investigated [2]) what figure of equilibrium exists of a rotating fluid mass, which arrives at the end of the stable series of JACOBIAN ellipsoids, he considered an infinitely near configuration by impressing upon the original surface a small variation expressed in ellipsoidal harmonics. With this variation correspond variations $\triangle m$, $\triangle \theta$ and $\triangle (U+V)$ of the total mass, moment of momentum and total energy. If the rotating mass is to change actually from the original into the new configuration, three conditions, viz.

$$\triangle m = 0, \quad \triangle \theta = 0, \quad \triangle (U+V) = 0 \ldots\ldots (13)$$

have to be satisfied. Moreover the natural course of evolution is designated by a fourth condition

$$\triangle U > 0 \text{ or } \triangle V < 0 \ldots\ldots\ldots (14)$$

The kinetic energy of the system increases at the cost of the potential energy.

What the equation of the figure of equilibrium was in this case, is in our case the equation giving the distribution of the density of the gas in the equatorial plane of the disc. It determines the structure of equilibrium completely and can be represented in the original state by

$$\varrho_e = \varrho_0 \, e^{-a r^{\frac{1}{2}}} \ldots\ldots\ldots (12)$$

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[1]) On the electrostatic field of the sun, due to its corpuscular rays. Proc. Kon. Akad. v. Wetensch., Amsterdam, **33**, 614—618 (1930). On the electrostatic field of the sun as a factor in the evolution of the planets. Proc. Kon. Akad. v. Wetensch., Amsterdam, **33**, 719—722 (1930).

[2]) Figures d'équilibre d'une masse fluide.

Let us now consider the slightly varied distribution

$$\varrho_e + \triangle\varrho_e = \varrho_0 (1+\varphi)\, e^{-(a+\psi) r^{\frac{1}{2}} - 2\gamma \sin b r^{\frac{1}{2}} + 2\delta \cos b r^{\frac{1}{2}}} \quad . \quad . \quad . \quad (15)$$

with

$$\varphi \ll 1, \quad \psi \ll a, \quad \gamma \ll 1, \quad \delta \ll 1 \quad . \quad . \quad . \quad . \quad . \quad (16)$$

or

$$\varrho_e + \triangle\varrho_e = \varrho_0\, e^{-a r^{\frac{1}{2}}} \{1 + \varphi - \psi r^{\frac{1}{2}} - 2\gamma \sin b r^{\frac{1}{2}} + 2\delta \cos b r^{\frac{1}{2}}\} \quad . \quad (17)$$

if developed up to first order small quantities.

In this new density-distribution a tendency to concentration into rings is anticipated. The anticipation is allowed, but we have to state why we try the functions $\sin br^{\frac{1}{2}}$ and $\cos br^{\frac{1}{2}}$ in stead of, for instance, $\sin br$ and $\cos br$. According to the theorem of minimum loss of energy, ϱ_e has to satisfy the equation

$$\frac{d \lg \varrho_e}{dr} + 2r \frac{d^2 \lg \varrho_e}{dr^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The solution of this equation, up to first order quantities was (12). Inserting (15) we get

$$\frac{d \lg \varrho_e}{dr} + 2r \frac{d^2 \lg \varrho_e}{dr^2} = b^2 \{\gamma \sin br^{\frac{1}{2}} - \delta \cos br^{\frac{1}{2}}\} \quad . \quad . \quad . \quad (19)$$

As γ and δ are small quantities, the right side remains small compared with both terms on the left side. Hence, (18) remains satisfied as far as first order quantities. As the power $r^{\frac{1}{2}}$ in $\sin r^{\frac{1}{2}}$ and $\cos r^{\frac{1}{2}}$ is the only power for which this holds, it is the only power which we are reasonably allowed to try. I may add that the formation of separate rings is the natural way in which the loss of energy by friction can decrease further and would even finally vanish when the rings rotate like rigid bodies.

It is a happy circumstance that $2 \sin br^{\frac{1}{2}}$ and $2 \cos br^{\frac{1}{2}}$ can be readily combined with the term

$$e^{-a r^{\frac{1}{2}}} \text{ to } e^{-(a \pm bi) r^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

We can integrate and are in the position to conclude that our 4 conditions are satisfied in certain cases. Hence, nature will actually follow the indicated course and generate a series of concentric rings.

It would have been preferable, if we had added in the exponent of (15) to the terms with $\sin br^{\frac{1}{2}}$ and $\cos br^{\frac{1}{2}}$ a sequence of terms adaptable to any arbitrary variation. This way is barred by mathematical difficulties and the author was glad to find that from the infinite variety of ways in which our disc might disintegrate it was apparently only free to take the one sanctioning our trial with the first terms.

If r_n and r_{n-1} are the radii of the n*th* and $(n-1)$*th* ring

$$\sqrt{r_n} - \sqrt{r_{n-1}} = \frac{2\pi}{b} = \text{constant} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

a formula destined to replace BODE's mysterious law, which has given rise to so much controversy.

Let us now calculate the variations $\triangle m$, $\triangle \theta$, $\triangle U$ and $\triangle V$.

$$\triangle m = (2\pi)^{\frac{3}{2}} \left(\frac{RT}{fM}\right)^{\frac{1}{2}} \int_0^\infty r^{\frac{5}{2}} \triangle \varrho_e \, dr \quad . \quad . \quad . \quad . \quad . \quad (22)$$

$$\triangle \theta = (2\pi)^{\frac{3}{2}} \left(\frac{RT}{fM}\right)^{\frac{1}{2}} \int_0^\infty r^{\frac{9}{2}} (\omega \triangle \varrho_e + \varrho_e \triangle \omega) \, dr \quad . \quad . \quad . \quad (23)$$

$$\triangle U = \tfrac{1}{2} (2\pi)^{\frac{3}{2}} \left(\frac{RT}{fM}\right)^{\frac{1}{2}} \int_0^\infty r^{\frac{9}{2}} (\omega^2 \triangle \varrho_e + \varrho_e \triangle \omega^2) \, dr \quad . \quad . \quad . \quad (24)$$

whereas

$$\triangle \varrho_e = \varrho_e \{\varphi - \psi r^{\frac{1}{2}} + 2\gamma \sin b r^{\frac{1}{2}} + 2\delta \cos b r^{\frac{1}{2}}\} \quad . \quad . \quad . \quad (25)$$

Moreover

$$\omega^2 = \frac{fM}{r^3} - \frac{RTa}{2r^{\frac{3}{2}}} \ldots\ldots \text{approximately} \quad . \quad . \quad . \quad . \quad . \quad (26)$$

$$\omega^2 + \triangle \omega^2 = \frac{fM}{r^3} + \frac{RT}{r} \frac{d}{dr} lg \, \varrho_e \left(1 + \frac{\triangle \varrho_e}{\varrho_e}\right) = \omega^2 + \frac{RT}{r} \frac{d}{dr} \frac{\triangle \varrho_e}{\varrho_e} \quad . \quad (27)$$

whence

$$\triangle \omega^2 = \frac{RT}{2r^{\frac{3}{2}}} \{-\psi - 2b\gamma \cos b r^{\frac{1}{2}} + 2b\delta \sin b r^{\frac{1}{2}}\} \quad . \quad . \quad . \quad (28)$$

$$\omega = \frac{(fM)^{\frac{1}{2}}}{r^{\frac{3}{2}}} - \frac{RTa}{4(fM)^{\frac{1}{2}}} \ldots\ldots \text{approximately} \quad . \quad . \quad . \quad . \quad . \quad (29)$$

$$\omega + \triangle \omega = \frac{(fM)^{\frac{1}{2}}}{r^{\frac{3}{2}}} + \frac{RT}{2(fM)^{\frac{1}{2}}} \frac{d}{dr} lg \, \varrho_e \left(1 + \frac{\triangle \varrho_e}{\varrho_e}\right) = \omega + \frac{RT}{2(fM)^{\frac{1}{2}}} \frac{d}{dr} \frac{\triangle \varrho_e}{\varrho_e} \quad (30)$$

whence

$$\triangle \omega = \frac{RT}{4(fM)^{\frac{1}{2}}} \{-\psi - 2b\gamma \cos b r^{\frac{1}{2}} + 2b\delta \sin b r^{\frac{1}{2}}\} \quad . \quad . \quad . \quad (31)$$

The potential energy V of the nebula consists of 2 parts. The first part, V_1, is the energy stored up in the attraction of the sun, or

$$V_1 = -4\pi fM \int_0^\infty \varrho_e \, dr \int_0^\infty e^{-\frac{fM}{2RT} \frac{h^2}{r^3}} \left\{1 + \left(\frac{h}{r}\right)^2\right\}^{-\frac{1}{2}} dh \quad . \quad . \quad (32)$$

To evaluate this integral, let us make use of

$$\left\{1 + \left(\frac{h}{r}\right)^2\right\}^{-\frac{1}{2}} = e^{-\frac{h^2}{2r^2}} \text{ approximately} \quad . \quad . \quad . \quad . \quad (33)$$

It follows that

$$V_1 = -(2\pi)^{\frac{3}{2}} (fMRT)^{\frac{1}{2}} \int_0^\infty \left(1 + \frac{RT}{fM} r\right)^{-\frac{1}{2}} r^{\frac{3}{2}} \varrho_e \, dr \, . \quad . \quad . \quad (34)$$

The variation of V_1 is

$$\left.\begin{aligned} \triangle V_1 = -(2\pi)^{\frac{3}{2}} (fMRT)^{\frac{1}{2}} \int_0^\infty r^{\frac{3}{2}} \\ e^{-ar^{\frac{1}{2}}} \{\varphi - \psi r^{\frac{1}{2}} + 2\gamma \sin b r^{\frac{1}{2}} + 2\delta \cos b r^{\frac{1}{2}}\} \, dr + \tfrac{1}{2} RT \triangle m \end{aligned}\right\} . \quad (35)$$

The second part of V is the potential energy of the gas of the nebula proper, V_2. We only need the variation $\triangle V_2$. We suppose the temperature not to vary. Hence the internal energy is constant and

$$\triangle V_2 = -\int\int\int p \triangle v \, dv \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

is the change of potential energy due to the density variations. This reduces to

$$\triangle V_2 = RT \int\int\int \triangle \varrho \, dv = RT \triangle m \quad . \quad . \quad . \quad . \quad (37)$$

However, as $\triangle m = 0$. V_2 and the second part of V_1 both vanish. Therefore

$$\triangle V = -(2\pi)^{\frac{3}{2}} (fMRT)^{\frac{1}{2}} \int_0^\infty r^{\frac{3}{2}} e^{-ar^{\frac{1}{2}}} \{\varphi - \psi r^{\frac{1}{2}} + 2\gamma \sin b r^{\frac{1}{2}} + 2\delta \cos b r^{\frac{1}{2}}\} \, dr. \quad (38)$$

We can now solve the equations

$$\triangle m = 0, \ \triangle \theta = 0, \ \triangle (U + V) = 0 \, . \quad . \quad . \quad . \quad . \quad (13)$$

The solution requires a rather painful calculation, which the author will reproduce here as concisely as is compatible with clearness.

If

$$\frac{a}{b} = x \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

we get

$$\left.\begin{aligned} \varphi - 7\frac{\psi}{a} + 2x^7 (x^2+1)^{-7} [\gamma \{-7x^6 + 35x^4 - 21x^2 + 1\} + \\ + \delta \{x^7 - 21x^5 + 35x^3 - 7x\}] = 0 \end{aligned}\right\} . \quad (40)$$

$$\left.\begin{aligned} \varphi - 8\frac{\psi}{a} + 2x^8 (x^2+1)^{-8} [\gamma \{-8x^7 + 56x^5 - 56x^3 + 8x\} + \\ + \delta \{x^8 - 28x^6 + 70x^4 - 28x^2 + 1\}] = \\ = -\frac{180\, RT}{fMa^2} [-\varphi + 10\frac{\psi}{a} + \text{ other term}] \backsim 0 \end{aligned}\right\} . \quad (41)$$

$$\left.\begin{aligned} \varphi - 5\,\frac{\psi}{a} + 2x^5(x^2+1)^{-5}[\gamma\{-5x^4+10x^2-1\}+\delta\{x^5-10x^3+5x\}] = \\ = \frac{105\,RT}{fMa^2}\left[-\varphi + 7\,\frac{\psi}{a} + \text{other term}\right] \backsim 0 \end{aligned}\right\} \quad (42)$$

The "other terms" in (41) and (42) are of the same kind, but more extensive than the third term on the left side of these equations. However, it is not necessary to specify them, because we will afterwards be able to show that the multiplicators before the square brackets on the right sides of (41) and (42) are constants of a magnitude between 10^{-3} and 10^{-4}. Therefore, it can make no serious difference in the solution, if we put the right sides $= 0$. Doing this we can easily eliminate φ and ψ and get

$$\frac{\gamma}{\delta} = \frac{3x^9 - 76x^7 + 154x^5 - 28x^3 - 5x}{23x^8 - 140x^6 + 98x^4 + 4x^2 - 1} \quad . \quad . \quad . \quad . \quad (43)$$

The density fluctuation being given by the exponent

$$2\,(-\gamma\ sin\ b\,r^{\frac{1}{2}} + \delta\ cos\ b\,r^{\frac{1}{2}}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

whereas

$$b = \frac{a}{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

we learn that equation (43) determines the "phase-angle"

$$arc\ tg\left(-\frac{\gamma}{\delta}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

in function of the "wavelength", or the "wavelength" in function of the "phase-angle". Actually, an initial compression, however small, along a circle with unknown radius r_m, will start the evolution on its new course, the nebula having reached a state of instable equilibrium. If then, by some cause or other the "phase-angle" of the fluctuation would be determined, we could solve x from

$$tg\,\frac{a\,r_m^{\frac{1}{2}}}{x} = -\frac{3x^9 - 76x^7 + 154x^5 - 28x^3 - 5x}{23x^8 - 140x^6 + 98x^4 + 4x^2 - 1} \quad . \quad . \quad . \quad (47)$$

x being determined, we can find the 3 relative proportions of the other 4 variables φ, ψ, γ and δ from (43) and any 2 of the 3 equations (40) (41) (42). A solution can always be found and there are as many sets of solutions as there are solutions of the equation (47).

(To be continued.)